APPENDIX A

Schrödinger Dynamics in the Non-Relativistic Limit of Scalar QED

Just as classical limits of non-relativistic quantum mechanics reproduce descriptions of classical dynamics, so too should a NR limit of a relativistic quantum theory reproduce dynamics which are valid in the NR quantum regime and are described by the Schrödinger equation. Using scalar quantum electrodynamics as a case study, we will show how Schrödinger dynamics emerge from taking NR limits of the equations of motion produced by a QFT.

A.1. Setting up Scalar QED

We begin with the Lagrangian for scalar QED:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger(D^\mu\phi) - m_\phi^2\phi^\dagger\phi \tag{A.1}$$

where ϕ is the complex-valued scalar field, and $F_{\mu\nu}$ is the electromagnetic field strength tensor. $D_\mu \equiv \partial_\mu + ieA_\mu$ is the gauge-covariant derivative which governs coupling between the scalar field and the electromagnetic field, with coupling strength e.

Any dynamics in the background electromagnetic field are governed by the $(F)^2$ term in Equation (A.1). In general, we would write the vector potential in two components as $A^\mu = A^\mu_{\text{static}} + A^\mu_{\text{dynamic}}$, but for simplicity in this construction we will choose for the background potential to be static and originating from a point-like and non-recoiling source, such as the nucleus of an atom. This allows A^0 to be treated classically, as if there were a charge Q with a relative mass $M/m_\phi \gg 1$ located at the origin.

Continuing with this construction, we will choose $V(\vec{r}) = \frac{kQ}{|\vec{r}|}$, which naturally comes with vector potential

$$A^\mu = (\frac{kQ}{c|\vec{r}|}, 0, 0, 0). \tag{A.2}$$

Since we have isolated the background field to have no dynamics of its own, and $A^\mu = \frac{kQ}{c|\vec{r}|}\delta^{\mu,0}$, we retrieve

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A_0 - \partial_\nu A_0 = 0,$$

and the Scalar QED Lagrangian reduces to

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - m_\phi^2\phi^*\phi. \tag{A.3}$$

A.2. The Field Equation and Background Coupling

Now that we have a reduced expression for the Lagrangian density for our system, we can vary the action with respect to ϕ^* and find the equations of motion. By the chain rule,

$$\begin{aligned}\mathcal{L} =&(D_\mu\phi)^*(D^\mu\phi) - m_\phi^2\phi^*\phi \\ =&\phi^*(D_\mu D^\mu\phi) - \phi^* m_\phi^2\phi \end{aligned} \tag{A.4}$$

which, varying the action with respect to ϕ^*, gives

$$\begin{aligned} 0 &= (\delta\phi^*)(D_\mu D^\mu\phi - m_\phi^2\phi) \\ 0 &= (D^2 - m_\phi^2)\phi \end{aligned} \tag{A.5}$$

Which are the equations of motion for the field ϕ. While it looks at a glance that the coulomb background field is gone, remember that we got rid of the $(F)^2$ term because the background field has no dynamics! The field is still very much present; it is in the definition of $D_/mu$, the gauge-covariant derivative

Now, carefully expanding the gauge-covariant derivatives,

$$D_\mu =\partial_\mu + i\frac{e}{\hbar}A_\mu$$

$$=\partial_\mu + i\frac{e}{\hbar}A_0 \tag{A.6}$$

and so

$$\begin{aligned} D^2 &= (\partial_\mu + i\frac{e}{\hbar}A^0)(\partial^\mu + i\frac{e}{\hbar}A^0) \\ D^2 &= \Box - (\frac{e}{\hbar})^2(A^0)^2 + 2i\frac{e}{\hbar}A^0\partial_0 \end{aligned} \tag{A.7}$$

where $\Box$ is the d'Alembert operator, $\Box \equiv \frac{1}{c^2}\partial_0^2 - \nabla^2$, and we have used the fact that the background field is time independent. We are left with:

$$0 = (\Box - (\frac{e}{\hbar})^2(A^0)^2 + 2i\frac{e}{\hbar}A^0\partial_0 - m_\phi^2)\phi \tag{A.8}$$

The Coulomb field enters the dynamics in the kinematic term, but because of the structure of D^2 it can be thought of separately.

A.3. Non-relativistic Limit Ansatz

In order to isolate the non-relativistic limit of the equations of motion, we introuce ansatz that separates the fast rest mass oscillation from the slower background dynamics:

$$\phi(t,\vec{r}) = e^{-imc^2\frac{t}{\hbar}}\psi(\vec{r},t) \tag{A.9}$$

It is natural to factor out $e^{imc^2\frac{t}{\hbar}}$, as this term represents the fast oscillations of the rest mass, functioning as a local wave packet for the electron. The function $\psi(\vec{r},t)$ describes the slower background dynamics, and acts as a guiding envelope for this wave packet. Thus, $\psi(\vec{r},t)$ will describe the non-relativistic dynamics of the system.
Computing the resulting time derivatives:

$$\begin{aligned} \partial_0\phi &= -\frac{ic^2m}{\hbar}e^{-\frac{ic^2mt}{\hbar}}\psi(\vec{r},t) + e^{-\frac{ic^2mt}{\hbar}}\partial_0\psi(\vec{r},t) \\ \partial_0^2\phi &= -\frac{c^4m^2}{\hbar^2}e^{-\frac{ic^2mt}{\hbar}}\psi(\vec{r},t) - \frac{2ic^2m}{\hbar}e^{-\frac{ic^2mt}{\hbar}}\partial_0\psi(\vec{r},t) + e^{-\frac{ic^2mt}{\hbar}}\partial_0^2\psi(\vec{r},t) \end{aligned} \tag{A.10}$$

and pluging into Equation (A.8):

$$\begin{aligned}0 = e^{-\frac{ic^2mt}{\hbar}}(&-c^2\nabla^2\psi(\vec{r},t) - \frac{(A^0)^2e^2}{\hbar^2}\psi(\vec{r},t) + \frac{2A^0ec^2m}{\hbar^2}\psi(\vec{r},t) \\ &+\frac{2i}{\hbar}\left(A^0e - c^2m\right)\partial_0\psi(\vec{r},t) + \partial_0^2\psi(\vec{r},t))\end{aligned} \tag{A.11}$$

We can multiply everything by $\frac{\hbar^2}{2mc^2}e^{\frac{ic^2mt}{\hbar}}$, before re-organizing by powers of c to return

$$\begin{aligned}i\hbar\partial_0\psi(\vec{r},t) = &-\frac{\hbar^2\nabla^2\psi(\vec{r},t)}{2m} + A^0e\psi(\vec{r},t) - \frac{(A^0)^2e^2}{2c^2m}\psi(\vec{r},t) \\ &+\frac{iA^0e\hbar}{c^2m}\partial_0\psi(\vec{r},t) + \frac{\hbar^2}{2c^2m}\partial_0^2\psi(\vec{r},t)\end{aligned} \tag{A.12}$$

where $i\hbar\partial_0\psi(\vec{r},t)$ has been moved to the left hand side of the equality.
Taking the non-relativistic limit of Equation (A.12), we send $c \to \infty$:

$$i\hbar\partial_0\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + A^0e\right)\psi(\vec{r},t) \tag{A.13}$$

What we are left with is an expression closely resembling the Schrödinger Equation - in fact, it is the Schrödinger Equation! This expression governs the time evolution of $\psi(\vec{r},t)$, which sees the effective potential $V_{\text{effective}}(\vec{r}) = eA^o$, the classical electric potential. Equation (A.13) is what we would have written if we had desired to build a description of this exact system in non-relativistic quantum mechanics.

Thus, beginning from the Lagrangian density of a scalar field coupled to an electromagnetic field, Schrödinger dynamics are recovered in the non-relativistic limit. By ignoring the leading relativistic terms, zeroth and first time derivatives of ψ mixed with A^0 have been neglected, and so some detail has been lost about how the fields couple at relativistic speeds of $\mathcal{O}(\frac{1}{c^2})$), as the solution is non-relativistic by construction.