APPENDIX B

Momentum Transfer in the Classical Limit of Scalar QED

In this Appendix, we will work through an example calculation to demonstrate how classical observables can be extracted from gauge-invariant observables calculated within a quantum field theory in the appropriate limits. Working within scalar QED, this derivation will calculate the impulse resulting from an interaction between two spinless scalar particles at the lowest order of perturbation theory, where they interact via a single photon exchange.

B.1. The Scattering Amplitude and Classical Impulse Formula

The scattering amplitude A, hereafter referred to as simply the amplitude, is an invariant expression with information about the scattering event, where $|A|^2$ =(Probability of interaction).

While an amplitude itself is not an observable (its square is, being a probability), they are related to potentials via Fourier transforms and contain information about other observables. Several observables can be calculated directly from an amplitude, one such example being the momentum transfer, or impulse, from an interaction. The relation is

$$\Delta p^{\mu} = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot b} \mathcal{A}(q) q^{\mu} (2\pi)\delta(q\cdot p). \tag{B.1}$$

The Equation (B.1) is the derivative of a Fourier transform of the amplitude, changing the argument from momentum transfer to impact parameter. The delta function $\delta(q\cdot p)$ extracts from the integral only values of q^{μ} for which $p\cdot q = 0$. In the rest frame of the light scalar, $p = (p^0, \vec{0})$, there should be no energy transfer, so only $q = (0, \vec{q})$-type momenta contribute. Thus energy conservation is enforced via this delta function. From here, the next step is to calculate the amplitude itself.

B.2. The Tree Level Amplitude

B.2.1. Building an Amplitude with Feynman Rules

In order to write down the amplitude for a given diagram in scalar QED, one follows the the Feynman rules [9]:

(1) Draw the diagram, labeling incoming, outgoing, and internal leg momenta.
(2) Write down a contributing factor for each vertex and internal propagator.
(3) conserve energy and momentum at each vertex.
(4) Integrate over internal momenta.
(5) Cancel any remaining delta functions, and multiply by i.

In Particular, the Feynman rules for scalar QED say that diagram components contribute to the amplitude like so:

$$\begin{aligned}\textbf{Photon Propagator} &= \frac{-ig^{\mu\nu}}{q^2 - i\epsilon} \\ \textbf{Scalar Propagator} &= \frac{i}{p^2 - m^2 + i\epsilon} \\ \textbf{3-point Vertex} &= ie(p + p')^\mu \\ \textbf{4-point Vertex} &= 2ie^2 g^{\mu\nu}\end{aligned}$$

B.2.2. Building the $2 \to 2$ Tree-Level Scalar QED Amplitude

Considering the case for a lightweight scalar (with incoming momentum p^μ and outgoing momentum $p^\mu + q^\mu$) scattering off a heavy scalar (with incoming and outgoing momentum

P^μ) via single photon exchange, the diagram will include one photon propagator and one vertex associated with each scalar. Conserving momentum at each vertex and taking the limit that the heavier of the two particles is non-recoiling results in figure B.1.

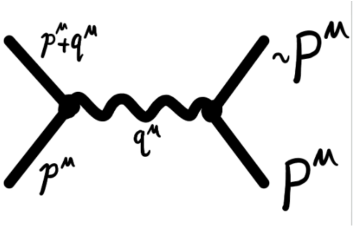

Figure B.1. Single photon exchange with a non-recoiling particle

Following the rules, the amplitude goes like:

$$\mathcal{A}(q) = ie(p+p+q)_\nu \frac{-ig^{\mu\nu}}{q^2 - i\epsilon} ie(P+P)_\mu = \frac{ie^2(2p^\mu + q^\mu)(2P_\mu)}{q^2 - i\epsilon}. \tag{B.2}$$

Multiplying by i,

$$\mathcal{A}(q) = \frac{-e^2(2p^\mu + q^\mu)(2P_\mu)}{q^2 - i\epsilon}. \tag{B.3}$$

In the limit where the heavier particle is non-recoiling, the numerator simplifies to

$$(2p^\mu + q^\mu)(2P_\mu) = 4(p^\mu P_\mu) + 2(q^\mu P_\mu) \tag{B.4}$$

and $P^\mu = (M, 0, 0, 0)$, where M is large. For a small momentum transfer $4(E_p M) \gg 2(q^0 M)$, and so the amplitude becomes

$$\mathcal{A}(q) = \frac{-e^2(2p^\mu)(2P_\mu)}{q^2 - i\epsilon}. \tag{B.5}$$

The next step is to insert this final amplitude expression into Equation (B.1).

B.3. The q^0 Integral

Taking p^μ to be on-shell and moving only in the $\hat{z}$ direction, it reduces to $p^\mu = (E/c, 0, 0, p^z)$. This results in the relation $E^2 = m^2c^4 + p_z^2c^2$, and, in the rest frame of the massive particle,

$P^\mu = (P^0, 0, 0, 0)$. The delta function in Equation (B.1), $\delta(q\cdot p)$, enforces that contributions to the integral come with $q^0 = 0$. After re-writing the expression from Equation (B.1) as

$$\Delta p^\mu = \int d^3q \int dq^0 \frac{1}{(2\pi)^4} e^{-iq\cdot b}\mathcal{A}(q) q^\mu (2\pi)\delta(q\cdot p) \tag{B.6}$$

the q^0 portion of the integral can be evaluated straightforwardly, as the delta function selects $q^0 = 0$ and reduces the entire expression to

$$\Delta p^\mu = \left(0, \int d^3q \frac{1}{(2\pi)^4} e^{-i\vec{q}\cdot\vec{b}}\mathcal{A}(q) q^i (2\pi)\delta(q^0)\right). \tag{B.7}$$

The remaining spatial integral describes momentum transfer in the transverse plane orthogonal to the trajectory.

B.4. Impulse in The Transverse Plane

The resulting momentum transfer is entirely transverse. This can be seen by considering

$$\Delta\vec{p} = \int d^3q \frac{\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{b}}\mathcal{A}(q)\delta(q\cdot p). \tag{B.8}$$

The impact parameter is transverse to trajectory, so $e^{-i\vec{q}\cdot\vec{b}}$ isolates $\vec{q}_\perp$. The amplitude is even in $|\vec{q}|^2$, so the integrand is constructed by multiplying an odd and even function in p^z, which naturally evaluates to zero. Therefore, the resulting impulse will be entirely transverse, which is to say $\Delta\vec{p}$ lies in the $x-y$ plane. Additionally, by $\delta(q\cdot p) = \delta(p^0q^0 - q^zp^z)$, only $p^0 = \frac{p^zq^z}{q^0}$ type-momenta will be relevant for the integration.

Following this through,

$$\begin{aligned}\Delta\vec{p} &= \int d^3q \frac{\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{b}} \left(\frac{-4e^2(P_0p^0)}{q^2 - i\epsilon}\right) \delta(p^0q^0 - q^zp^z) \\ &= \int d^3q \frac{\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{b}} \left(\frac{-4e^2(P_0p^0)}{q^2 - i\epsilon}\right) \frac{1}{p^z}\delta(q^z - \frac{q^0}{v})\end{aligned} \tag{B.9}$$

Using the delta function to isolate $q^z = \frac{q^0}{v}$, the integral over z can be separated from that over x and y, defining the transverse momentum transfer $\vec{p}_\perp = (p_x, p_y)$ and likewise the transverse impact parameter $\vec{b}_\perp = (b_x, b_y)$. The expression above becomes

$$\Delta\vec{p}_\perp = \frac{-4e^2(P_0 p^0)}{|p^z|} e^{-i\frac{q^0}{v}b^z} \int d^2q \frac{1}{(2\pi)^3} \frac{(q^x, q^y, q^0/v)}{q^2 - i\epsilon} e^{-i\vec{b}_\perp \cdot \vec{p}_\perp} \tag{B.10}$$

where remaining integral is clearly identifiable as a Fourier transform. Simplifying one step further, we can write $m^2 \equiv (q^0/v)^2 - i\epsilon$,

$$\Delta\vec{p}_\perp = \frac{-4e^2(P_0 p^0)}{|p^z|} e^{-i\frac{q^0}{v}b^z} \int d^2q \frac{1}{(2\pi)^3} \frac{\vec{q}_\perp}{|\vec{q}_\perp|^2 - m^2} e^{-i\vec{b}_\perp \cdot \vec{p}_\perp} \tag{B.11}$$

This is evaluated using a Fourier transform identity. In the large-$|\vec{b}_\perp|$ limit,

$$\Delta\vec{p}_\perp = \frac{-e^2}{v} \frac{\vec{b}_\perp}{|\vec{b}_\perp|^2}. \tag{B.12}$$

Up to prefactors, this is identical to the classical result for the impulse imparted by the coulomb deflection of a charged particle moving past a fixed charge; the reliance on charge, velocity, and impact parameter is identical.

This is an important result, demonstrative of the validity of this investigation. We are genuinely able to move from gauge and Lorentz-invariant QFT expressions, such as scattering amplitudes, to classically relevant observables by taking appropriate limits and carrying out the correct calculations.