APPENDIX C

Ladder Resummation and the Lippman-Schwinger Equation

In Appendix B, the example calculation of the impulse from an amplitude expression was done at first order of perturbation theory where two massless scalars interacted via a single photon exchange in scalar QED.

In this section, we investigate the effect of additional particle exchanges. This will be done by continuing to work within the framework of scalar QED, beginning with the one-loop ladder diagram before considering the limit in infinte photon exchanges.

C.1. One-Loop Ladder

The one-loop ladder, or "box diagram," includes two photon exchanges. Each scalar particle propagates freely between these exchanges.

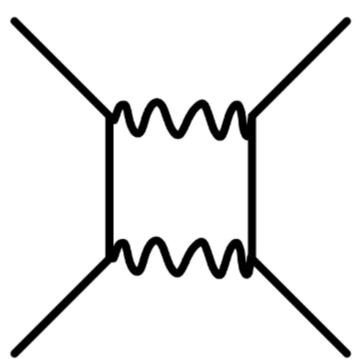

Figure C.1. The Box Diagram

The amplitude for this construction is, up to prefactors,

$$\mathcal{A} \approx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{D_1 D_2 D_3 D_4}. \tag{C.1}$$

D_i terms are contributions from propagators of the box diagram, two scalars and two exchanged photons. The rules for writing them are the same as Equation (B.2). The propagator of the less massive scalar contributes a term that goes like

$$\frac{i}{(p-l)^2 - m^2 + i\epsilon}. \tag{C.2}$$

This can be expanded in the non-relativistic limit by inserting $E \approx m + \frac{|\vec{p}|^2}{2m}$ and taking the limit that $p^2 \approx l^2 \ll m^2$. By expanding the squares, discarding terms appropriately, using $E - m \approx \frac{\vec{p}^2}{2m} = E_{NR}$, and ignoring rest mass contributions to the energy in the non relativistic limit this gives the following expression for the propagator:

$$\frac{i}{E - \ell^0 - \frac{(\vec{p}-\vec{\ell})^2}{2m} + i\epsilon}. \tag{C.3}$$

This expression is a Green's function for time evolution in the presence of an interaction. E is the energy of the particle prior to the collision, and ℓ^0 describes the change in energy from the collision. We could think of this as being of the form

$$\frac{i}{E - H + i\epsilon} \tag{C.4}$$

where H is the Hamiltonian of the system, $H = \frac{(\vec{p}-\vec{\ell})^2}{2m} + \ell^0$.

C.2. Higher Loop Orders

Extrapolating how contributions for higher loop orders of ladder diagrams not difficult to do, we simply attach more and more propagators with the correct momenta, as in C.2.

When writing the expression for the amplitude for this construction, each additional scalar propagator will contribute like:

$$\frac{i}{E - \sum_{i=1}^{n} \ell_i^0 - \frac{\left(\vec{p} - \sum_{i=1}^{n} \vec{\ell}_i\right)^2}{2m} + i\epsilon}. \tag{C.5}$$

In Equation (C.5), each intermediate scalar propagator appears as an independent factor. For a ladder diagram with k exchanged photons, rather than combining into a single denominator the contributions multiply. The scalar propagation structure is therefore

$$\prod_{n=1}^{k} \left(\frac{i}{E - \sum\limits_{i=1}^{n} \ell_i^0 - \frac{1}{2m} \left(\vec{p} - \sum\limits_{i=1}^{n} \vec{\ell}_i\right)^2 + i\epsilon} \right). \tag{C.6}$$

This multiplication of Green's function factors G_0 is what produces the geometric series structure for T which we will see in a moment, since inserting an additional photon exchange corresponds to inserting an additional factor of $G_0 V$.

C.3. Towards Re-Summation

From the modern on-shell viewpoint, the ladder diagrams represent the repeated exchange of the mediating quantum (here, the photon), which amounts to iterating the classical potential between successive scattering events. Each insertion of V corresponds to a single quantum exchange, and each G_0 corresponds to on-shell propagation between interactions.

Diagrammatically, this follows C.2. Resumming all such exchanges naturally generates

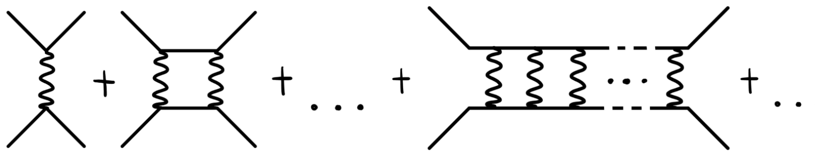

Figure C.2. Building towards the limit of infinite photon exchanges

$$T = V + VG_0V + VG_0VG_0V + \cdots = \frac{V}{1 - G_0V}, \tag{C.7}$$

whereach iteration of V is a photon exchange, and each G_0 is a time-evolution governing Green's function identified in Equation (C.5) describing free propagation between exchanges. This espression is the geometric series leading to the Lippmann–Schwinger equation,

$$T = V + VG_0T. \tag{C.8}$$

In this way, Lippmann–Schwinger is revealed directly as the resummed on-shell physics of infinitely repeated long-range interactions, and the bound-state poles of T emerge from the condition $1 - G_0V = 0$, which is precisely where the scattering amplitude develops physical poles associated with classical bound states. This illustrates one of the central advantages of the amplitudes framework: classical observables appear as reorganizations or limits of the same on-shell building blocks.

C.3.1. Lippman-Schwinger equation

Written out fully, Lippman-Schwinger equation goes like

$$T(\vec{p}, \vec{p}') = V(\vec{p}, \vec{p}') + \int \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{p}, \vec{q}) \frac{1}{E - \frac{\vec{q}^2}{2m} + i\epsilon} T(\vec{q}, \vec{p}') \tag{C.9}$$

In this expression:

- $\vec{p}$ is the 3-momentum of the incoming scalar particle

- $\vec{p}'$ is the 3-momentum of the outgoing scalar particle
- $\vec{q}$, the variable of integration, is the 3-momentum of the particle after interacting with the potential. In Equation (C.3), it is the term $(\vec{p} - \vec{\ell})^2$
- E is the total energy. In Equation (C.3), it is the term $E - \ell^0$

$V =$ Fourier Transform$(A_{2\to 2})$ is the potential from the the single photon exchange amplitude. From the tree-level single photon exchange amplitude in scalar QED,

$$A_{2\to 2}(q) = -\frac{4e^2(P\cdot p)}{q^2 + i\epsilon}, \tag{C.10}$$

and working in the rest frame of the heavy particle, $P^\mu = (M, \vec{0})$, the on-shell condition enforces $q^0 = 0$. Thus the potential is obtained by a three-dimensional Fourier transform,

$$V(\vec{q}) = -\frac{4e^2 M E_p}{\vec{q}^2}, \qquad \vec{q} \equiv \vec{p}' - \vec{p}. \tag{C.11}$$

Equivalently, in position space,

$$V(\vec{r}) = \int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{q}) = -\frac{e^2 M}{\pi^2}\frac{1}{|\vec{r}|}, \tag{C.12}$$

up to conventional choices for Fourier normalization. This reproduces the expected Coulomb potential, consistent with the classical impulse derived in Appendix B.