APPENDIX D

Radiated Momentum from Classical Scattering

Now we will compute leading-order momentum radiated by the electromagnetic waves during the scattering process investigated in earlier sections. The framework remains scalar QED, and the setup of the lightweight scalar particle scattering off of a heavy, non-recoiling particle is the same. The difference is that now there will be an additional on-shell photon with momentum k^μ and polarization $\varepsilon^\mu(k)$ radiated during the scattering process. We take p^μ and p'^μ to be the incoming and outgoing momentum of the lightweight particle, respectively.

D.0.1. Leading Radiation Amplitude

At tree level, there are four diagrams that contribute to this 5-point scattering. Two in which the photon is emitted by the lightweight particle either before or after scattering, which we will call diagrams 1 and 2, and two in which the photon is emitted by the heavy

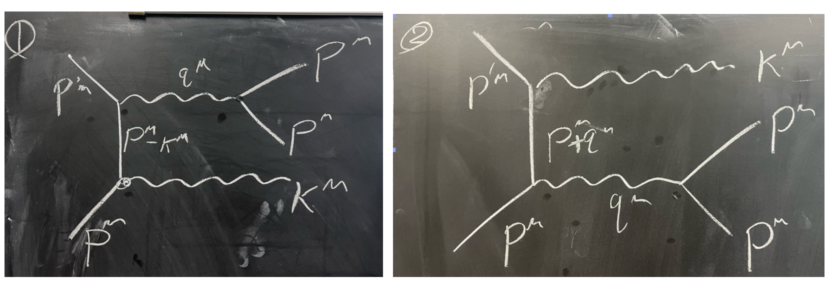

Figure D.1. Diagrams 1 and 2

particle, either before or after scattering, which we will call diagrams 3 and 4. Notice that the internal structure with the photon propagator is precisely what we saw in the $2 \to 2$

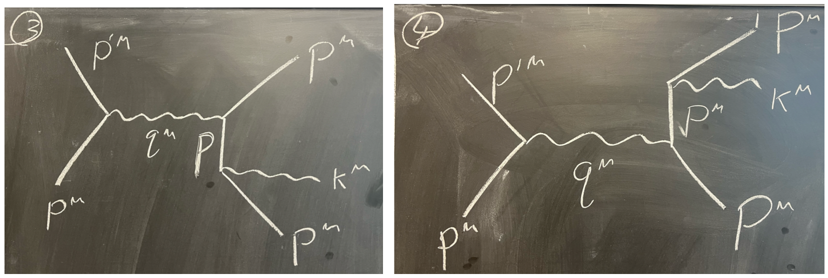

Figure D.2. Diagrams 3 and 4

scattering case earlier, now with an additional scalar propagator attached. Following the Feynman rules we wrote earlier, we can write the amplitudes for each diagram and sum.

Following the Feynman rules, the individual diagrams contribute like so:

$$\mathcal{A}_1(p,p',k) \sim -4e^3 \frac{[(2p-k)^\mu \varepsilon_\mu][(p-k)^\nu P_\nu]}{((p-k)^2 - m^2 - i\epsilon)(q^2 - i\epsilon)} \tag{D.1}$$

$$\mathcal{A}_2(p,p',k) \sim -4e^3 \frac{[(2p+2q-k)^\mu \varepsilon_\mu][p^\nu P_\nu]}{((p+q)^2 - m^2 - i\epsilon)(q^2 - i\epsilon)} \tag{D.2}$$

$$\mathcal{A}_{3,4}(p,p',k) \sim -8e^3 \frac{[P^\mu \varepsilon_\mu][p^\nu P_\nu]}{q^2 - i\epsilon} \tag{D.3}$$

Diagrams 3 and 4 contribute the same, since we have already taken the limit that the heavy scalar is both non-recoiling and massive. We can further simplify the denominators of these amplitudes by taking that, for diagrams 1 and 2, $q = p' - p + k$. By enforcing both that the radiated photon is on shell, $k^2 = 0$, and the Ward identity, $k \cdot \varepsilon(k) = 0$, we can write the sum of these terms as

$$\mathcal{A}_1(p,p',k) \sim e \frac{-e^2(2p^\mu)(2P_\mu)}{q^2 - i\epsilon} \left(\frac{p\cdot\varepsilon}{p\cdot k} \right) = e \left(\frac{p\cdot\varepsilon}{p\cdot k} \right) \mathcal{A}_{\text{el}} \tag{D.4}$$

$$\mathcal{A}_2(p,p',k) \sim e \frac{-e^2(2p^\mu)(2P_\mu)}{q^2 - i\epsilon} \left(\frac{P\cdot k}{p\cdot k} - \frac{p'\cdot\varepsilon}{p'\cdot k} \right) = e \left(\frac{P\cdot k}{p\cdot k} - \frac{p'\cdot\varepsilon}{p'\cdot k} \right) \mathcal{A}_{\text{el}} \tag{D.5}$$

$$\mathcal{A}_{3,4}(p,p',k) \sim e \frac{-e^2(2p^\mu)(2P_\mu)}{q^2 - i\epsilon} (2P\cdot\varepsilon) = 2e\,(P\cdot\varepsilon)\,\mathcal{A}_{\text{el}} \tag{D.6}$$

where $\mathcal{A}_{\text{el}}$ is the elastic scattering amplitude computed earlier. The full amplitude then goes like:

$$\mathcal{A}_{\text{rad}}(p, p', k) \sim e \left(\frac{p \cdot \varepsilon}{p \cdot k} + \frac{P \cdot k}{p \cdot k} - \frac{p' \cdot \varepsilon}{p' \cdot k} + 2P \cdot \varepsilon \right) \mathcal{A}_{\text{el}} \tag{D.7}$$

Taking the “soft" limit of the emitted photon ($k^\mu \ll p^\mu$) we return that

$$\mathcal{A}_{\text{rad}}(p, p', k) \sim e \left(\frac{p \cdot \varepsilon}{p \cdot k} - \frac{p' \cdot \varepsilon}{p' \cdot k} \right) \mathcal{A}_{\text{el}} \tag{D.8}$$