APPENDIX E

Emergent Bound States in the T-Matrix

If the T-matrix develops a pole at energy E_n, we might write

$$T(E) \sim \frac{|\psi_n\rangle \langle\psi_n|}{E - E_n + i\epsilon} \tag{E.1}$$

Physically, we interpret this pole to mean that when $E = E_n$ there is no scattering. That is, the particles enter a bound state. $|\psi_n\rangle$ must be normalizable, since when we calculate an amplitude $\mathcal{A} \sim \langle\varphi| T |\varphi\rangle$ we need to be able to use $|\varphi\rangle = \sum c_i |\psi_i\rangle$ to extract $\langle\psi_n|\psi_n\rangle = 1$. This is a physical bound state! Because it's physical, it must be normalizable.

This directly implies that the particle does not change state as it time evolves, as can be seen in Eq. Equation (E.1) where the ket-bra in the numerator isolates the same state on either side. When calculating an amplitude there will be a singularity, since there is no scattering!

E.0.1. Simplifying Near Pole

Plugging in Equation (E.1) to the Lippman-Schwinger equation, Equation (C.8),

$$\frac{|\psi_n\rangle \langle\psi_n|}{E - E_n + i\epsilon} \sim V + VG_0(E)\frac{|\psi_n\rangle \langle\psi_n|}{E - E_n + i\epsilon} \tag{E.2}$$

Taking this to be near a pole,

$$V \ll \lim_{E \to E_n} T(E).$$

We can also multiply both sides by $E - E_n + i\epsilon$, and act with an arbitrary, normalized state $|\varphi\rangle = \sum c_i |\psi_i\rangle$ for which $c_n \neq 0$. Dividing out c_n, we are left with

$$|\psi_n\rangle = G_0(E_n)V |\psi_n\rangle \tag{E.3}$$

near the pole. This is a homogeneous integral equation satisfied by the bound-state wave function, $|\psi_n\rangle$. The "operator" acting on $|\psi_n\rangle$ is an integral operator, looking back to Equation (C.9).

Looking at the derivation of the Lippman-Schwinger equation beginning from the Schrödinger equation, we see that $(H_0 + V)\,|\psi_n\rangle = E_n\,|\psi_n\rangle$ was the origin of the derivation. Therefore, we take E_n to be the energy eigenvalue of the state $|\psi_n\rangle$ in the presence of the potential V, which we recall is single photon exchange. Interestingly, by re-summing, going to the Lippman-Schwinger equation, and looking near a pole at bound states we have uncovered good old Schrödinger dynamics! In particular, Eq. Equation (E.3) encodes Schrödinger dynamics in *integral* form, rather than the usual differential form.