APPENDIX G

Partial Wave Decomposition and Form-Factor Matching

In this section we transition from plane-wave descriptions of radiation to a basis that explicitly diagonalizes total angular momentum and helicity.

G.1. Review - From Spherical Harmonics to Helicity Waves

Here we will set up the covariant partial-wave formalism by working through how angular momentum and helicity interact. The goal is to see how scalar and vector radiation modes decompose into spherical waves, and how this leads to spin-weighted spherical harmonics.

G.1.1. Scalar Radiation Patterns

Considering a scalar wave emitted with a definite energy ω. In a chosen frame, the spatial field can be decomposed like so:

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} j_\ell(\omega r) Y_{\ell m}(\hat{r}) \tag{G.1}$$

where $Y_{\ell m}(\hat{r})$ are the ordinary spherical harmonics, which form a complete orthonormal basis for angular functions. Since we are considering a scalar field, we're not losing out on any information by choosing this basis (for a different field, we would need a different choice - more on this in a moment). Another feature of the spherical harmonics is how they behave under rotations. As the vector $\hat{r} \to \hat{r}'$, any given spherical harmonic is now expressed as a linear combination of spherical harmonics of the same degree. That is,

$$Y_{\ell m}(\hat{r}') = \sum_{m'=-\ell}^{\ell} A_{mm'} Y_{\ell m'}(\hat{r}). \tag{G.2}$$

where $A_{mm'}$ is a matrix of order $(2\ell - 1)$ that depends on the rotation (it comes from taking the complex conjugate of an element of the Wigner-D matrix).

Really, what we're doing here is using the spin-weighted spherical harmonics to form our angular basis and making the choice that $s = 0$, for which the spin-weighted spherical harmonics are identical to the typical spherical harmonics. The spin-weighted spherical harmonics ${}^{s}Y_{\ell m}$, by the way, are $U(1)$ gauge fields that carry additional $U(1)$ symmetry and are denoted by an extra parameter, s. As noted above, ${}^{0}Y_{\ell m} = Y_{\ell m}$.

G.1.2. Vector Fields on a Sphere

We'll now consider a classically radiated vector field $A^i(\vec{r})$, such as the electric field of a photon, at large r. At each point on a sphere of radius r, the field can be decomposed into a radial component lying along $\hat{r}$ and two transverse components in the plane perpendicular to $\hat{r}$.

These two transverse components are particularly important for describing the physical polarizations of massless vector particles, such as the photon, because they are all that survive! A photon has two physical polarization states, and these are encoded in these transverse components. Further, since $A^i(\vec{r})$ is a gauge field, we can always choose a gauge such that the radial component vanishes but we cannot do this for the transverse components.

Let's consider a local coordinate system at $\vec{r}$ such that the transverse plane is spanned by two basis vectors, $\vec{e}_1$ and $\vec{e}_2$. The most general transverse vector is then written $\vec{A}_\perp = A_1\vec{e}_1 + A_2\vec{e}_2$. Rotating this vector about $\hat{r}$ by some angle α, the new vector is $\vec{A}'_\perp =$

$[A_1 \cos(\alpha) - A_2 \sin(\alpha)]\, \vec{e}_1 + [A_1 \sin(\alpha) + A_2 \cos(\alpha)]\, \vec{e}_2$. If we define the quantity $A_\pm = A_1 \pm iA_2$, then

$$\begin{aligned} A'_\pm &= [A_1 \cos(\alpha) - A_2 \sin(\alpha)] \pm i\, [A_1 \sin(\alpha) + A_2 \cos(\alpha)] \\ &= A_1\, [\cos(\alpha) \pm i \sin(\alpha)] + iA_2\, [-i \sin(\alpha) \mp \cos(\alpha)] \\ &= A_1\, [\cos(\alpha) \pm i \sin(\alpha)] \pm iA_2\, [\cos(\alpha) \pm i \sin(\alpha)] \\ &= e^{\pm i\alpha} A_\pm \end{aligned} \tag{G.3}$$

which is to say that $A_\pm$ transforms with a phase $e^{\pm i\alpha}$ under the rotation by α about $\hat{r}$. Physically, this phase corresponds to the photon's helicity ($h = \pm 1$) and is imposed onto the two components of the photon's circular polarization, A_1 and A_2. This space is an example of a *vector bundle.*

G.1.3. Defining Spin Weight

A field $f(\hat{r})$ on the sphere has spin weight s if it transforms under rotations by angle α about the local normal $\hat{r}$ as:

$$f(\hat{r}) \to e^{-is\alpha} f(\hat{r}). \tag{G.4}$$

In literature, some sign conventions for s differ. This motivates defining *spin-weighted spherical harmonics* (mentioned above) ${}^sY_{\ell m}$, which have several interesting properties:

- ${}^sY_{\ell m}$ are eigen functions of the total orbital angular momentum operators L^2 and L_z with eigenvalues $\ell(\ell + 1)$ and m.
- ${}^sY_{\ell m}$ transform under rotations with spin weight, as defined above.
- For a massless particle with helicity h, spin weight $s = -h$ is often the case.

- $\ell \geq |s|$ is required, since it ensures that ${}^{s}Y_{\ell m}$ remain well defined and nontrivial. ℓ must be large enough to contain the spin structure.

G.1.4. Photon Plane Wave Decomposition

A single photon plane wave state $\left|\vec{k}, h\right\rangle$ has 3-momentum $\vec{k}$, energy $\omega = |\vec{k}|$, and definite helicity $h = \pm 1$. This is analogous to decomposing a sound wave into it's harmonics, or analyzing the multi-pole content of emitted radiation – in this case, we're looking at constituent angular momentum content.This state can be expanded in terms of states with definite total angular momentum j and projection m. Schematically, this looks like

$$\left|\vec{k}, h\right\rangle = \sum_{j=|h|}^{\infty} \sum_{m=-j}^{j} C_{jm}(\hat{k}, h) \left|\omega, j, m, h\right\rangle \tag{G.5}$$

where $C_{jm}(\hat{k}, h)$ are coefficients given by the spin-weighted spherical harmonics ${}^{-h}Y_{jm}(\hat{k})$ where $\hat{k} = \vec{k}/\omega$.

The explicit expansion can be rewritten in terms of the spin-weighted spherical harmonics. To do this, the key will be rotating the $|\omega, j, m, h\rangle$ state towards the direction of $\hat{k}$. We might think of the coefficients, then, as accomplishing this rotation for each given state of definite angular momentum j and m. This will be simple to write in terms of the spin-weighted spherical harmonics, as they are directly related to the Wigner rotation matrices. the explicit expansion is

$$\left|\vec{k}, h\right\rangle = \sum_{j=|h|}^{\infty} \sum_{m=-j}^{j} \sqrt{\frac{4\pi}{2j+1}} {}^{-h}Y_{jm}(\hat{k}) \left|\omega, j, m, h\right\rangle . \tag{G.6}$$

If $h \to 0$, as for a scalar particle, this reduces to

$$\left|\vec{k}, 0\right\rangle = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sqrt{\frac{4\pi}{2j+1}} Y_{jm}(\hat{k}) \left|\omega, j, m, 0\right\rangle , \tag{G.7}$$

which is an expansion only in terms of the typical spherical harmonics.

G.1.5. Defining Covariant Angular Momentum Helicity States $|\omega, j, m, h\rangle$

The goal is to define a basis for single massless particle states that have definite energy, total angular momentum, z-component of angular momentum, and helicity.

We'll begin with a massless particle plane wave $\left|\vec{k}, h\right\rangle$, which is an eigen state of the 3-momentum $\vec{P}$ and helicity $\hat{h}$ operators, with eigenvalues $\vec{k}$ and h respectively. Generally speaking, 3-momentum and total angular momentum are not simultaneously diagonalizeable, and so this state is generally not an eigen state of J^2.

To define the eigen states of J^2 and J_z, we'll need to choose a frame. We'll choose a fixed, future-directed timelike 4-vector u^μ. In a specific lab frame, we might have $u^\mu = (1, 0, 0, 0)$. The energy of the particle in the frame of this vector is $\omega = k \cdot u$.

We define $|\omega, j, m, h\rangle$ to be eigen states, such that:

- $(P \cdot u) |\omega, j, m, h\rangle = \omega |\omega, j, m, h\rangle$
- $J^2 |\omega, j, m, h\rangle = j(j+1) |\omega, j, m, h\rangle$
- $J_Z^{(u)} |\omega, j, m, h\rangle = m |\omega, j, m, h\rangle$
- $\hat{h} |\omega, j, m, h\rangle = h |\omega, j, m, h\rangle$

Note that $J_Z^{(u)}$ is simply J_z defined with respect to the axis of the u^μ frame. These states are normalized, and form a complete basis for single particles of fixed helicity h:

$$\mathbb{I}_h = \sum_{j=|h|}^{\infty} \sum_{m=-j}^{j} \int_0^\infty \frac{d\omega}{2\omega(2\pi)^3} |\omega, j, m, h\rangle \langle \omega, j, m, h| \tag{G.8}$$

Of course, this completeness relation reinforces that all states $\left|\vec{k}, h\right\rangle$ can be described by a linear combination of states $|\omega, j, m, h\rangle$.

Up to normalization and delta functions, the spin-weighted spherical harmonics are defined by the overlap between a plane wave state and an angular momentum helicity state:

$$\left\langle \vec{k}, h' | \omega, j, m, h \middle| \vec{k}, h' | \omega, j, m, h \right\rangle = C \cdot \delta_{hh'} \cdot \delta(\omega_k - \omega) \cdot {}^{-h}Y_{jm}(\hat{k}_u) \tag{G.9}$$

where $\omega_k = |\vec{k}|$, and $\hat{k}_u$ is the direction of $\vec{k}$ in the rest frame defined by u^μ, and C is a normalization factor. We can also write ${}^{-h}Y_{jm}(k;u)$, where k is the 4-momentum.

Recall that ${}^{-h}Y_{jm}(k;u)$ can be constructed via the Winger D-matrices, they can also be built from combinations of vector spherical harmonics or by acting on Y_{jm} with raising/lowering operators built from derivatives.

G.1.6. Orthonormality

We can express orthonormality in a Lorentz invariant setting, before moving to the frame defined by u^μ. The Lorentz invariant expression looks like:

$$2\omega \int d^4k\delta_+(k^2)\delta(k\cdot u-\omega)({}^{-h}Y^*_{j'm'}(k;u))({}^{-h}Y_{jm}(k;u)) = \delta_{jj'}\delta_{mm'}. \tag{G.10}$$

By the identity $d^4k\delta_+(k^2) = d^3k/(2|\vec{k}|)$ and isolating $\omega = |\vec{k}|$ by $\delta(k\cdot u-\omega)$, which is to say restricting to the frame defined by u^μ, this expression becomes

$$\int d\Omega_{\hat{k}_u}({}^{-h}Y^*_{j'm'}(\hat{k}_u))({}^{-h}Y_{jm}(\hat{k}_u)) = \delta_{jj'}\delta_{mm'}. \tag{G.11}$$

Physically, the orthonormality of the spin-weighted spherical harmonics in the context of radiation patterns means that any radiation pattern is expressible as a linear combination of spin-weighted spherical harmonics.

G.2. Projecting Transition Amplitudes onto the Helicity Basis

Now we will apply this formalism to the physical process of photon emission from bound state transitions. Let $\mathcal{M}_{fi}(\vec{k},h_k) = \langle n'\ell'm'|\, J^\mu(0)\, |n\ell m\rangle\, \varepsilon^*_\mu(\vec{k},h_k)$ be the amplitude for the transition $|n\ell m\rangle \to |n'\ell'm'\rangle$ with the emission of a photon of momentum $\vec{k}$, energy $\omega = |\vec{k}|$, and helicity h_k. The 4-vector u^μ can be taken as the 4-velocity of the initial (heavy) bound state.

G.2.1. Defining Partial Wave Amplitudes (Form Factors)

We project the amplitude on to the angular momentum helicity basis for the outgoing photon. The partial wave amplitude (also known as the form factor) $F_j^{n\ell m\to n'\ell' m'}(h_k;\omega)$ is defined as:

$$F_j^{n\ell m\to n'\ell' m'}(h_k;\omega)\delta_{mm_{ph}}\cdot(\text{Normalizing Factor})=\int d\Omega_{\hat{k}_u}({}^{-h}Y^*_{jm_{ph}}(\hat{k}_u))\mathcal{M}_{fi}(\vec{k},h_k), \tag{G.12}$$

where $\omega_k = k\cdot u$ is fixed and m_{ph} is the photon's J_z projection. The form factor depends on j, helicity h_k, energy ω_k, and the initial/final bound states. By energy conservation, $\omega_k = E_{n\ell} - E_{n'\ell'}$.

This projection useful because it isolates the angular momentum content of $\mathcal{M}_{fi}(\vec{k},h_k)$ to a specific angular momentum channel j. This is to say that, physically, this form factor is the amplitude for emitting a photon which carries precisely j units of angular momentum. It contains the same information as $\mathcal{M}_{fi}$ (Of course it does, it's a projection!) but organized in such a way that angular momentum conservation is made manifest. Since we will soon be interested in tracking angular momentum closely, using the form factor will be convince. m_{ph} is the projection of the angular momentum of the particle along the axis of u^μ. In this way, $F_j^{n\ell m\to n'\ell' m'}(h_k;\omega)$ is the amplitude of emission for a photon of total angular momentum j and J_z eigenvalue m.

G.2.2. Application to the $2p \to 1s$ Transition

Now we will work through a concrete example of this projection, the $2p \to 1s$ transition mediated by the electromagnetic current. In Appendix F Equation (F.24), $\langle 100|\, J^z(0)\,|210\rangle$ was calculated explicitly. Inorder to accomplish this calculation, we will need to identify all matrix elements for $\vec{J}(0)\cdot\vec{\varepsilon}^*(\vec{k},h_k)$. Luckily, $J^x, J^y = 0$ for these transitions, and so we are left with

$$\mathcal{M}_{fi}(\vec{k},h_k) = \left\langle n'\ell'm' \right| J^z(0) \left| n\ell m \right\rangle \varepsilon_z^*(\vec{k},h_k) = \frac{4\,e}{27\,i\,a_0\,m}\sqrt{\frac{2}{3\pi}}\varepsilon_z^*(\vec{k},h_k). \tag{G.13}$$

The general circular polarization vector is

$$\vec{\varepsilon}(\vec{k},h_k=\pm 1) = \frac{1}{\sqrt{2}}(\hat{\theta} \pm i\hat{\phi}) \tag{G.14}$$

but, since only the $\hat{z}$ component will contribute, we are left with

$$\varepsilon_z^*(\vec{k},h_k) = \vec{\varepsilon}^*(\vec{k},h_k)\cdot\hat{z} = -\frac{\sin\theta}{\sqrt{2}}. \tag{G.15}$$

combining these results, the full amplitude is

$$\mathcal{M}_{2p\to 1s}(\vec{k},h_k) = i\frac{4\,e\,\sin(\theta)}{27\,a_0\,m}\sqrt{\frac{1}{3\pi}}. \tag{G.16}$$

Since this hydrogenic system is transitioning from $|210\rangle \to |100\rangle$, we can take advantage of angular momentum conservation to provide the expected non-zero values of j and h_k for this E1 transition. In particular, the photon must carry $j=1$ and $m=0$, with $h_k=\pm 1$. We can see this by observing the form of formfactordef. In order for the delta function to be nonzero, we require $m_{ph}=m=0$. $h_k=\pm 1$ is determined by the photon being on-shell.

The $j=1$ condition is fixed by the physical constraint of angular momentum conservation which requires that $J_{\text{total}} = L_{\text{bound}} + j_{\text{photon}}$. This plainly dictates that $j=1$ is required for the photon by the conservation law. Additionally, the $j=1$ condition can be seen from the projection integral. Beginning from $\mathcal{M} \propto \sin(\theta)$, we could ask how much of this amplitude corresponds to photons carrying $j=0,1,2,...$ and so on. We know that a function like sine is pure $j=1$ function, and whatever it is we will be able to write it as a combination of the $j=1$ spherical harmonics. Due to the orthogonality of the spherical harmonics, we will find that the projection is identically zero for all $j\neq 1$. It happens naturally!

For the dominant value $j=1$, we can project onto the form factor:

$$F_1^{210\to 100}(\pm 1;\omega) \sim \int d\Omega_{\hat{k}_u} ({}^{\mp 1}Y_{10}^*(\hat{k}_u)) i \frac{4\,e\,\sin(\theta)}{27\,a_0\,m}\sqrt{\frac{1}{3\pi}} \tag{G.17}$$

where

$${}^{\mp 1}Y_{10}^*(\theta,\phi) = \pm\sqrt{\frac{3}{8\pi}}\sin\theta$$

and so

$$\begin{aligned} F_1^{210\to 100}(\pm 1;\omega) &\sim \pm\, i \frac{e\sqrt{2}}{27\,\pi\,a_0\,m}\int d\Omega_{\hat{k}_u}\sin^2(\theta) \\ &= \pm\, i \frac{e\sqrt{2}}{27\,\pi\,a_0\,m}\int\int d\theta\, d\phi\, \sin^3(\theta) \\ &= \pm\, i \frac{8\,e\sqrt{2}}{81\,a_0\,m} \end{aligned} \tag{G.18}$$

Voilà!

G.3. Matching QFT and EFT in the Helicity basis – a Conceptual Discussion

Back in Appendix F, the QFT calculation of $\langle J^\mu \rangle$ was matched to an EFT operator. That same matching procedure can be similarly applied at the level of partial wave amplitude. In much the same way as before, this is done simply by calculating the amplitude $\mathcal{M}_{fi}^{\mathrm{EFT}}$ using EFT operators and then projecting onto the spin-weighted spherical harmonics, resulting in $F_j^{\mathrm{EFT}}(h_k;\omega_k)$.

One new consideration for this calculation would be the structure of the Wilson Coefficients. While at first order they were simply numbers, at higher order the Wilson Coefficients of the EFT will need to depend on $j, \omega_k,$ and h_k in order for the form factor EFT to fully match the QFT form factor for all cases. This matching of the angular structure is how the EFT might be designed to systematically capture the full angular and energy dependence of the full QFT.

G.4. Applications to Gravity

This covariant partial wave basis is powerful in that it makes angular momentum conservation manifest, handles relativistic kinematics naturally, and is adaptable to particles of any spin!

Appendix I considers the three-body scattering/decay processes, such as $A \to B + C + D$, or $A + B \to C + D$ towards the goal of applying these ideas to astrophysical black hole binaries. Since this covariant partial wave basis manifestly conserves angular momentum, one wold be able to calculate gravitational radiation from astrophysical binaries without needing to search for selection rules for j of the graviton - just by conserving angular momentum! Further, since this is adaptable for any spin, the spin-2 graviton will be handled smoothly within this formalism.