APPENDIX H

Bound State Effective Field Theories

Now we will begin to integrate what we've done so far to address more complex, multi-particle phenomena and push towards astrophysical applications. In this section, we will:

- Apply EFT principles to describe bound state formation with radiative emission, incorporating particle structure.
- Explore connections between single-quantum emissions and classical radiation waveforms.
- Prepare to analyze three-body interactions and their classical limits.

H.1. EFT for Radiative Capture with Particle Structure

We will consdier the radiative capture process in which two initially free scalar particles A and B form a bound state AB and emit a photon γ. The particles have masses m_a , m_b and charges e_A , e_B, and AB will have principal quantum number n, orbital angular momentum L, etc. The process is

$$A + B \to (AB)_{nL} + \gamma \tag{H.1}$$

We will build an EFT for this process, assuming that A is not point-like but has some internal structure affecting its coupling to photons – this will be described by an electromagnetic form factor $F_A(q^2)$ where q is the photon momentum. Particle B is point-like, for simplicity.

H.2. Degrees of Freedom and EFT Lagrangian Sketch

We'll begin to sketch out the Lagrangian structure of the EFT. It's relevant low-energy degrees of freedom should be fields for A, B, $(AB)_{nL}$, and the photon A_μ. The full lagrangian should include:

- Kinetic terms for the free A, B, $\Psi_{(AB)}$, and A_μ
- An interaction term that couples A and B to $\Psi_{(AB)}$. This is responsible for the binding, its Wilson coefficient would be related to bound state properties found from ladder re-summation. Call the interaction term $g_{form}\phi_A\phi_B\Psi^\dagger_{(AB)} + h.c.$
- Interaction terms for A and B coupling to the photon A_μ.

We have

$$\begin{aligned}\mathcal{L}_{\text{EFT}} \sim &[\phi_A \text{ Kinetic}] + [\phi_B \text{ Kinetic}] + [\Psi_{(AB)} \text{ Kinetic}] + [A_\mu \text{ Kinetic}] + \\ &[\phi_A \Leftrightarrow \gamma \text{ coupling}] + [\phi_B \Leftrightarrow \gamma \text{ coupling}] + [\phi_A\phi_B \Leftrightarrow \Psi_{(AB)} \text{ coupling}] + [\text{mass terms}]\end{aligned} \tag{H.2}$$

But what do these terms look like? Let's sketch it out. Schematically, a general kinematic term goes like $\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi$. The interaction terms for A and B to $\Psi_{(AB)}$ is described above, and the interaction term governing A and B coupling to the photon A_μ will be handled by an EFT operator:

$$\mathcal{L}_{\text{EFT}} \sim D_{A\,\mu}\phi^*_A D^\mu_A\phi_A + D_{B\,\mu}\phi^*_B D^\mu_B\phi_B + \frac{1}{2}\partial_\mu\Psi^\dagger_{(AB)}\partial^\mu\Psi_{(AB)} - \frac{1}{4}(F_{\mu\nu})^2 \tag{H.3}$$

$$\begin{aligned}&+ m_A\,\phi^*_A\phi_A + m_B\,\phi^*_B\phi_B + (m_A + m_B)\,\Psi^\dagger_{(AB)}\Psi_{(AB)} \\ &+ g_{form}\phi_A\phi_B\Psi^\dagger_{(AB)} + g^*_{form}\phi^*_A\phi^*_B\Psi_{(AB)}\end{aligned} \tag{H.4}$$

where D^μ_A and D^μ_B are the gauge covariant derivatives defined in guagecovdiv, which describe kinematics of fields A and B as well as coupling to the photon field A_μ via coupling $e_A F_A(q^2)$ and e_B, respectively.

Before we treated with the electromagnetic current related to a point particle, a term which went like $J^\mu \sim ie(\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)^\dagger\phi)$. Now that we have allowed A to have internal structure, this current needs to be modified to include the form factor $F_A(q^2)$ which describes the coupling of A's internal structure to the photons.

In momentum space (all together now) derivatives become momenta! so let's replace the derivatives with the appropriate momenta, and allow scaling by the form factor coupling. Now, $J^\mu_A(p,p') \sim e_A(p+p')^\mu F_A(q^2)$ where p and p' and the momenta of the incoming and outgoing scalar A, respectively.

We could also have built this by considering the contribution of the vertex $AA^*\gamma$ to the amplitude according to the Feynman rules – we just include the form factor for scaling since the coupling will be proportional to the form factor (instead of simply e_A).

H.3. Amplitude for Radiative Capture $A + B \to (AB) + \gamma$

We will now calculate the tree-level amplitude for this process in our EFT, taking the photon to be emitted from particle A. In this case, there is one relevant Feynman diagram at tree level:

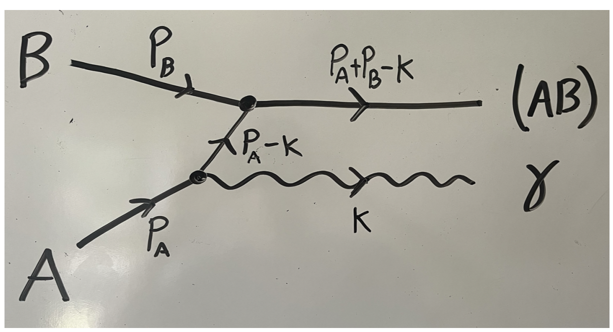

Figure H.1. Radiative Capture

From here, we will sketch out the form of the amplitude $\mathcal{M}$. schematically, we expect it to have the following form:

$$\mathcal{M} \sim [\text{Internal A Propagator}] \times [AA^*\gamma\text{Vertex}] \times [g_{form}(AB)\ \text{Vertex}]$$
$$\times[\text{Outgoing } (AB)] \times [\text{Photon Polarization } \varepsilon^*_\mu(k)] \qquad \text{(H.5)}$$

In many ways these contributions will mirror those we saw long ago for standard scalar QED, the only differences will be that the $AA^*\gamma$ vertex now carries a contribution from the form factor $F(k^2 = 0)$, and there is a new vertex with distinct coupling: $A_{internal} + B \to (AB)$. We then can sketch

$$\mathcal{M} \sim \frac{e_A F_A(0)(2p_A - k)^\mu \varepsilon_\mu(k)}{(p-k)^2 - m_A^2 + i\epsilon} * g_{form}\widetilde{\psi}_{nL}(\vec{p}_{rel}) \qquad \text{(H.6)}$$

where $p_{rel} = p_a + p_b - k$. We can see how the momentum-space wavefunction enters through the integral form of the vertex contribution to the amplitude:

$$\int d^4x \phi_{A_{internal}}(x)\phi_B(x)\Psi_{(AB)}(x) = \int d^4x\, e^{i\, p_{rel}\, x}\, \psi_{nL}(x) \to \widetilde{\psi}_{nL}(\vec{p}_{rel}) \qquad \text{(H.7)}$$

where we have taken the incoming scalar particles to be plane waves, and applied the appropriate definition for p_{rel}. Identifying this as a Fourier transform for the wavefunction of the bound state, we have essentially “projected" the $A_{internal} + B$ state onto the final bound state. *

H.4. Impact of the Form Factor $F_A(q^2)$

Let us assume a simple dipole form factor for A: $F_A(q^2) = 1/(1-q^2/\Lambda^2)^2$, where Λ is a scale related to the size of particle A. For a real emitted photon $q^2 = 0$, and so $F_A(0) = 1$ and coupling is entirely described by the charge e_A - we might say that e_A is the “full charge" of A.

If instead the photon were virtual, perhaps if (AB) were formed by $A + B$ exchanging a virtual photon, we will have $F_A(q^2 \neq 0)$. In this case, the interaction strength falls off

*Got great help from Adrian on this part from its maximum value of 1 as q increases. This is to say that for this dipole structure $F_A(q^2 \neq 0)$ results in weaker coupling than $F_A(q^2 = 0)$.

For the case of real photon emission in $A + B \to (AB) + \gamma$, the internal structure scale Λ might still influence higher-order terms or process, despite the fact that $F_A(0) = 1$. Such processes may initially have the photon virtuality as non-zero before becoming on-shell. The "size" of particle A is related to $1/\Lambda$.

H.5. Partial Wave Decomposition (Conceptual Discussion)

To decompose and interpret the angular distribution of the emitted photon γ, we can use formfactordef to project the amplitude of this emission on to the basis of spin-weighted spherical harmonics. This would also allow us to use the same selection rules as before to predict which angular momentum quantum numbers j for the photon we expect to dominate based on what is known about (AB) and $A + B$. For example, consider if (AB) is formed in an S-wave ($L = 0$) state and the $A + B$ system is also (predominantly) S-wave ($L = 0$) at low energies, $j = 1$ will dominate for the emitted photon.