APPENDIX I

Three-Body Dynamics in an Effective Field Theory

In the previous section we modeled bound states and their interactions with massless radiation. In the next step, we will model particle interactions with other *massive* particles. In particular, in this section we will develop and use an EFT to describe processes like binary breakup and rearrangement.

I.1. Matching: Fixing the Contact Interaction c_{AB}

Let's recall how a single contact interaction is matched. In our three-body EFT, the vertex $c_{AB}(\phi_A^\dagger \phi_A)(\phi_B^\dagger \phi_B)$ will represent the effect of a heavy mediator σ being "integrated out." Practically, this means that in the full UV theory the ferryman diagram for this interaction looks like this:

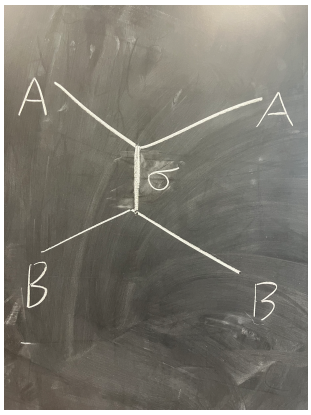

Figure I.1. $A + B \to A + B$ interaction with heavy mediator

where σ is a heavy mediator. In our EFT, we will treat this interaction as point-like, and the diagram becomes becomes:

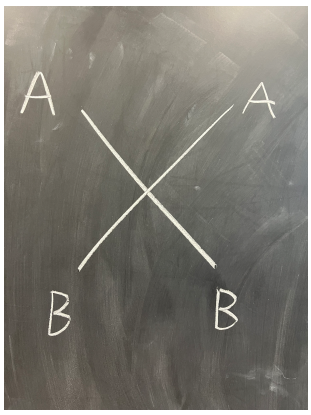

Figure I.2. $A + B \rightarrow A + B$ interaction in the EFT

To identify the Wilson coefficients c_{AB}, one will need to:

(1) Calculate the interaction in the UV theory. This will involve a mediator for σ, which will go like $\sim g_A\, g_B\, /(q^2 - m_\sigma^2)$ in the amplitude.
(2) Take the low-energy limit, $m_\sigma^2 >> q^2$. In this limit, the propagator term will become $\sim g_A\, g_B\, /(m_\sigma^2)$, which is indeed dimensionless.
(3) Calculate the interaction in the EFT. This is just the expression above with c_{AB} waiting to be solved for.
(4) Equate these results, identifying the structure of the Wilson coefficients c_{AB}. In particular, we see that $c_{AB} \sim g_A\, g_B\, /(m_\sigma^2)$ This is how the EFT “knows" about the underlying physics.

I.2. The Breakup and Rearrangement Processes - Schematic

Having outlined this matching principle, we can now explore the 3-body process. We will consider two processes in this section:

- Breakup: $A + (BC) \to A + B + C$ (meditated by the c_{AB} vertex)
- Rearrangement: $A + (BC) \to (AC) + B$ (meditated by particle C exchange)

Feynman diagrams for these processes are:

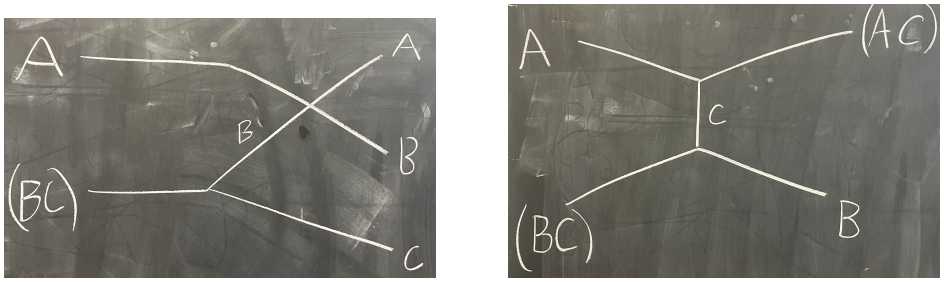

Figure I.3. The breakup (Left) and rearrangement (right) processes in Feynman diagram form

Both of the diagrams in Figure I.3 involve a "formation/breakup" vertex, of the form $B + C \to (BC)$ or $(BC) \to B + C$. In the Lagrangian, this take the form $g_{form}\Psi^{\dagger}_{BC}\phi_B\phi_C$. To make the EFT predictive, we must now determine this coupling g_{form}.

I.3. Calculation: Fixing the Formation Coupling g_{form}

We will fix g_{form} by ensuring the EFT correctly reproduces the properties of the bound state it represents. Recalling that poles of scattering amplitudes describe bound states, we will match the pole of the $B + C \to B + C$ scattering amplitude in the full theory and in the EFT. For simplicity, we will consider the ground state of (BC) which is an s-wave $(L = 0)$ bound state.

I.3.1. The EFT Pole

In our EFT, the $B + C \to B + C$ process can proceed via the s-channel exchange of the Ψ_{BC} particle.

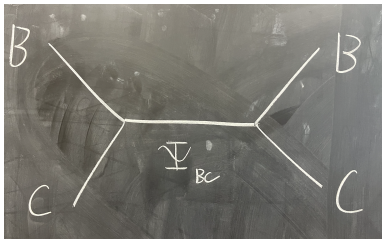

Figure I.4. $B + C \to B + C$ scattering via s-channel Ψ_{BC} exchange.

The amplitude for this process is built from one particle mediator and two interaction vertices, each with coupling g_{form}, and is expressed as

$$\mathcal{M}_{EFT} \sim \frac{|g_{form}|^2}{E_{cm}^2 - M_{BC}^2 + i\epsilon} \to \frac{|g_{form}|^2}{E_{cm} - M_{BC}} \tag{I.1}$$

in the limit that $E_{cm} \to M_{BC}$, and E_{cm} is the center of mass energy. Absorbing normalization factors into g_{form}, the residue of this pole is simply $|g_{form}|^2$.

I.3.2. The Non-Relativistic Quantum Mechanical Pole

In the full theory, which is described by non-relativistic QM with a potential, the residue of the S-matrix pole for an S-wave bound stat is directly related to the wavefunction at the origin. The relation is

$$\text{Residue} = -8\pi\mu\,|\widetilde{\psi}_{1s}(0)|^2 \tag{I.2}$$

where μ is the reduced mass of the BC system and $\widetilde{\psi}_{1s}(0)$ is the normalized momentum-space wavefunction at zero momentum. For a hydrogenic ground state, the momentum-space wavefunction is $\widetilde{\psi}_{1s}(\vec{p}) = 8\pi\sqrt{\pi a_0^3}\,/\,(1 + a_0^2\,|\vec{p}|^2)^2$, where a_0 is the Bohr radius.

I.3.3. The Matching Calculation

From the given formula, $\widetilde{\psi}_{1s}(0) = 8\pi\sqrt{\pi a_0^3}\,/\,(1 + a_0^2)^2$. From here it is straightforward to equate the residue from the EFT with the residue from NRQM to solve for $|g_{form}|$, which

is precisely what we shall now do. We begin from

$$\text{Residue} = -8\pi\mu\,|\frac{8\pi\sqrt{\pi a_0^3}}{(1+a_0^2)^2}|^2 = |g_{form}|^2 \tag{I.3}$$

which implies

$$|g_{form}| = \frac{8\pi^2\sqrt{8\,\mu\,a_0^3}}{(1+a_0^2)^2}. \tag{I.4}$$

I.3.4. Physical Interpretation

Notice that the coupling of the bound state to its constituents g_{form} is proportional to the wavefunction at the origin. Why does this make physical sense? It implies that bound states whose wavefunctions have higher values near their center of mass are more strongly coupled to the individual particles, while those with more distributed wavefunctions are less so. We might take the coupling strength to tell us how sensitive the bound state is to the state of the constituent scalars, and vice versa. In particular, for states with high angular momentum ($L > 0$) where the wave function may be equal or near to zero at the origin, there is very weak coupling between the constituent particles and the bound state which implies less formation probability.