APPENDIX J

Recoil Kinematics

The emission of each graviton (or photon) causes the system to recoil as momentum is carried away. While often ignored, this is a fundamental aspect of radiation and momentum conservation.

Let's consider a bound state of mass M at rest, decaying to a state of mass M_f by emitting a single massless particle of energy ω_k, such as a graviton or a photon. Using 4-momentum conservation, and the fact that this particle is on-shell, we know that $k^2 = 0$, which is to say $\omega_k^2 - |\vec{k}|^2 = 0$. The 4-momenta are:

$$
\begin{aligned}
p_M &= (M, \vec{0}) \\
p_{M_f} &= (E_f, -\vec{k}) \\
q &= (\omega_k, \vec{k})
\end{aligned}
\tag{J.1}
$$

where $E_f = \sqrt{M_f^2 + |\vec{k}|^2} = \sqrt{M_f^2 + \omega_k^2}$. We can use 4-momentum conservation to write

$$
\begin{aligned}
\omega_k &= M - \sqrt{M_f^2 + \omega_k^2}. \\
M - \omega_k &= \sqrt{M_f^2 + \omega_k^2}
\end{aligned}
\tag{J.2}
$$

Solving for the energy of the photon, we get

$$\omega_k = \frac{M^2 - M_f^2}{2M}. \tag{J.3}$$

Applying $\Delta E = M - M_f$:

$$\omega_k = \frac{(M + M_f)(M - M_f)}{2M} = \frac{\Delta E(2M - \Delta E)}{2M} = \Delta E(1 - \frac{\Delta E}{2M}) \tag{J.4}$$

Taking $\Delta E << M$, we can expand about $\frac{\Delta E}{2M}$ and the expression becomes

$$\omega_k = \Delta E \left(1 - \frac{\Delta E}{2M} + \mathcal{O}\left(\frac{\Delta E^2}{M^2} \right) \right) . \tag{J.5}$$

The total energy radiated by a binary inspiral is a significant fraction of the rest mass. For a binary of $10\,M_\odot$ black holes, roughly 5% of the total mass is radiated as GWs. If recoil were neglected, the total fractional error would be about $\approx \Delta E/2M = 4.75\%$ error.

We might compare the existence of this fractional error in the energy transfer to the inclusion of dissipative terms in the $\mathcal{PN}$ expansion, such as the 2.5 $\mathcal{PN}$ term.