CHAPTER 1

Introduction

There has been a great deal of excitement following advancements made in gravitational wave astronomy throughout the past decade. In 2015, the LIGO experiment first detected the gravitational waves resulting from a binary black hole merger [1], thus verifying a century old prediction of general relativity. With additional projects such as LISA on the horizon [8] the long-term futures of gravitational wave astronomy and multi-messenger astrophysics seem promising.

Numerical relativity methods have long been employed in conjunction with detection as a method for predicting and interpreting the data from gravitational wave observatories [2]. These approaches use tools such as the Post-Newtonian, $\mathcal{PN}$, expansion to find approximate solutions to Einstein's equations and describe dynamics. Some famous uses of the $\mathcal{PN}$ expansion have been calculating the precession of mercury and solving the two-body problem in the context of general relativity.

In a recent paper from Northwestern University's Center for Interdisciplinary Exploration and Research in Astrophysics [10], these methods were use to model the problem of three black holes scattering in a stellar cluster and to analyze the resulting gravitational wave. The authors found that a detectable phase shift exists in the gravitational waveform caused by a third body scattering with a black hole binary, which could be used to interpret dynamics from gravitational wave data.

Of interest to this project is the problem of simulating these chaotic interactions, "chaotic," in this context, implies a high sensitivity to initial conditions and interaction forces. Many current implementations use the Post-Newtonian ($\mathcal{PN}$) expansion to approximate general relativity, which typically is expressed in the code as a series of pairwise forces truncated at a certain order.

The goal of this project is to use analytical methods in a quantum field theory (QFT) description of the gravitational interaction to provide tools for computational astrophysicists to improve their simulations, and to demonstrate that this method provides physical information unique from that in the $\mathcal{PN}$ expansion. One such potentially unique piece of information comes from the fact that GR allows gravity to self-interact, meaning there can be nontrivial 3-body gravitational interactions which cannot be characterized as a sum of pairwise interactions; the gravity mediating the interaction is self-interacting. Since the system is sensitive to initial conditions and interaction strengths, even a small true 3-body force could change the outcomes of these encounters dramatically. This, in turn, would directly alter the expected phase shift which is a direct result of separation between the third body and the binary. Additionally, this approach might provide a higher-resolution description of dynamics accounted for approximately in the $\mathcal{PN}$ expansion, such as the recoil from an emitted gravitational wave. A discussion of this particular feature can be found in Appendix J.

We will proceed by first reviewing the analytical tools to be used in this QFT description of the interaction, namely unitarity methods and the double copy. After presenting these ideas along with example calculations at tree level, we will move on to calculating the 6-point gravitational scattering amplitude for 3 massive scalar particles, extracting classical observables from this expression. The conclusion will review our findings, and note promising future directions for this work.

CHAPTER 2

Background

A quantum field theory, or QFT, provides a relativistic quantum mechanical description of a theory and its dynamics. In a QFT description of a theory particles are cast as excitations of a quantum field which may be coupled to other quantum fields, and the dynamics of this system can be studied by, for instance, calculating equations of motion from the theory's Lagrangian.

Importantly, a QFT description of a system should be consistent with classical and non-relativistic quantum mechanical descriptions of the same system in the appropriate limits. Just as classical limits of non-relativistic quantum mechanics reproduce descriptions of classical dynamics, so too does a NR limit of a relativistic quantum theory reproduce dynamics which are valid in the NR quantum regime. Appendix A includes an example calculation where Schrödinger dynamics are recovered in the non-relativistic limit of scalar quantum electrodynamics. Appendix C further codifies the relationship between QFT descriptions of dynamics and Schrödinger dynamics, where the Schrödinger equation is recovered in integral form as the Lippman-Schwinger equation.

Since astrophysical black holes are not quantum mechanical on the length scale of their Schwarzschild radii, we will need to take these limits in order to produce predictive expressions for their behavior beginning from an observable obtained in a QFT description of gravity. In this context, the black holes will be modeled by spinless scalar particles with mass and coupling to the gravitational field. In this sense, since the black holes are non-astrophysical (lacking traits such as charge and spin), the framework for this discussion is an effective field theory in which the scalar particle approximation for the black hole is valid.

The quantity we will search for in a QFT description of gravity is the scattering amplitude, a gauge and Lorentz invariant expression whose square describes the probability of a particular scattering event as particles evolve in time from $t = -\infty \rightarrow t = +\infty$. While the amplitude itself is not directly observable its square is, and the amplitude itself is directly connected to several classical observables when the appropriate limits are taken. An example calculation in the framework of scalar QED is provided in Appendix B, as well as a presentation of the traditional method for identifying amplitudes by following the Feynman rules for the relevant theory.

While the traditional path of following the Feynman rules to construct the gravity amplitude, such as is done for the scalar QED amplitude in Appendix B, is still valid, there are complications that come with this strategy. Off-shell Feynman rules for gravitons are complicated and include contact terms at every multiplicity of interaction, and the amplitude itself includes a factorially increasing number of local graphs as loop-level and multiplicity increase. Unitarity methods address the issue of complex off-shell graviton contributions, and the factorially increasing number of graphs is simplified by leveraging the duality between color and kinematics via the Double Copy [7].

2.1. Unitarity Methods

Unitarity methods are the modern technique for constructing higher-loop order gravity amplitudes, following the program of "sewing" together physical, on-shell quantities to construct amplitudes with both higher loop order and multiplicity. Thus, these unitarity cuts allow us to systematically construct and verify integrands of complicated interactions via on-shell, tree-level expressions [5].

2.1.1. Example Calculation: 4-Point Tree-Level Scalar QCD Amplitude

As an example calculation, we can construct the 4-point Tree-Level amplitude for scalar QCD via unitarity methods. For an interaction this simple using unitarity methods is indeed overkill, but this calculation will give a schematic by which we may address more complicated interactions later on.

To accomplish this, we consider two 3-point tree-level scalar QCD amplitudes and sew together the gluonic legs to form the four-point construction, as can be seen in Figure 2.1.

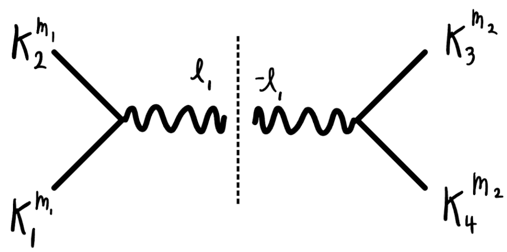

Figure 2.1. Sewing 3-point trees to form a 4-point amplitude

The key idea is that the gluonic legs are taken to be on-shell and to have the same momenta up to a negative sign. Since we are combining cubic graphs this results in a single 4-point topology: the s-channel.

Tn QCD, the kinematic numerator for the cubic interaction is

$$n_{3,1}^{\mathrm{QCD}}(k_1^{m_1}, k_2^{m_1}, \ell_1) = (k_1^{m_1} \cdot \varepsilon_\ell) \tag{2.1}$$

where ε_ℓ is the polarization of the gluon. For $n_{i,j}^{\mathrm{theory}}$, $i =$ the number of external legs, $j =$ the number of distinct massive scalar particles, and the theory label describes which theory the numerator belongs to (e.g.QCD, QED, gravity). Since the gluon is on-shell, $\ell_1^2 = 0$.

To sew together these amplitudes, we take the product of their numerators and use the projection operator

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{q^\nu p^\mu - p^\mu q^\nu}{p \cdot q}, \tag{2.2}$$

where q is an arbitrary, null reference momenta which will factor out for any physical expression if on-shell cut conditions and momentum conservation are properly enforced. To obtain the full 4-point tree-level numerator we will need to both use this projector and sub over all distinct leg permutations, in this case leg orderings $(1,2,3,4)$ and $(3,4,1,2)$. Doing so, and applying $(k_1^{m_1})^2 = (k_2^{m_1})^2 = m_1^2$, we find that

$$n_{4,2}^{\mathrm{QCD}}(k_1^{m_1}, k_2^{m_1}, k_3^{m_2}, k_4^{m_2}) = -\frac{1}{2}(m_1^2 + k_1^{m_1}\ (k_2^{m_1} + 2\, k_3^{m_2}) \tag{2.3}$$

which is indeed the correct kinematic numerator, in agreement with the literature [6].

2.2. Duality between Color and Kinematics and the Double Copy Formalism

The color-kinematics duality and Double Copy formalism address the second complication for calculating amplitudes in a gravity theory, the factorially increasing number of graphs. The core of the double copy recipe is the color-kinematics duality, which is the idea that representations exist where the kinematic numerator terms for gauge theory amplitudes are related by the same algebra as the generic color weights [3]. In such a representation the dynamics and charges are interchangeable, and graviton scattering can be realized by replacing the color weight with a second iteration of the kinematic numerator term [7]. This greatly reduces the number of graphs that need to be considered, and allows gravity amplitudes to be computed directly from gauge theory amplitudes.

2.2.1. The 2-Body Classical Gravitational Potential from a One-Loop Amplitude

As a benchmark, we will work through the derivation of the 2-body classical gravitational potential beginning from a tree-level gravity scattering amplitude, leveraging both unitarity methods and the double copy along the way and following from [7], [6], [11], and [4].

In section 2.1.1 the 4-point tree-level scalar QCD kinematic numerator for two distinct massive scalars was identified in Equation (2.3). By the double copy, this expression can be transformed into the numerator for $\mathcal{N}=0$ supergravity, since the $\mathcal{N}=0$ supergravity numerator is the square of the QCD numerator [6].

$$n_{4,2}^{\mathcal{N}=0}(1,2,3,4) = (n_{4,2}^{\mathrm{QCD}}(1,2,3,4))^2 = \frac{1}{4}(k_1 \cdot k_2 + 2\, k_1 \cdot k_3 + m_A^2)^2 \tag{2.4}$$

This is the double copy formalism at work. In order to update this numerator function for pure-gravity exchange, the extra-state dimension $\Delta_{4,2} = \frac{m_A^2 m_B^2}{D_s - 2}$ [6], where D_s is the number of space-time dimensions arising from the trace of the metric, needs to be subtracted away. The full amplitude for 2-to-2 scattering at tree level in Einstein-Hilbert gravity is found to be:

$$\mathcal{M}_{4,2}^{\mathrm{tree}}(1^{m_A}, 2^{m_A}, 3^{m_B}, 4^{m_B}) \sim \frac{\frac{1}{4}(k_1 \cdot k_2 + 2\, k_2 \cdot k_3 + m_A^2)^2 - \frac{m_A^2 m_B^2}{D_s - 2}}{(k_1 + k_2)^2} \tag{2.5}$$

where $(k_1 + k_2)^2 = q^2$, the momentum of the propagator, and $D_s = 4$.

Potentials are related to scattering amplitudes via Fourier transforms, and so to reproduce the classical Newtonian potential from this amplitude we first take the non-relativistic limit and apply Fourier transform. Takign the NR limit for this expression requires the realization of $|\vec{k}_i| << m_i$, and $k_i^0 \approx m_i + \frac{|\vec{p}_i|^2}{2m_i}$. Applying this to the amplitude, the NR limit of the 4-point tree-level Einstein-Hilbert gravity amplitude is identified.

$$\mathcal{M}_{4,2}^{\mathrm{NR\ tree}}(q^2) \sim \frac{m_A^2(m_A + m_B)^2 - \frac{1}{2} m_A^2 m_B^2}{q^2} \tag{2.6}$$

Simplifying this, we need only employ a Fourier transform carrying ourselves from momentum to position space to identify the potential. Resulting in

$$\mathcal{M}_{4,2}^{\mathrm{NR\ tree}}(q^2) \sim G \frac{m_A^2 m_B^2}{q^2} \ \rightarrow \ V_{\mathrm{Newton}}(r) = -G \frac{m_A m_B}{r}, \tag{2.7}$$

which is indeed the classical Newtonian potential.

CHAPTER 3

Calculation

3.1. The Research Goal

The purpose of this Master's Thesis is to study how the classical and non relativistic limits of observables in QFTs can be used to provide numerical astrophysics simulations with more precise computational tools which are directly connected to the underlying physics. The final goal will be to use QFT methods to calculate the Eikonal Phase for 3-body black hole gravitational scattering, which serves a generating function for other classical observables, in connection with the work done in [10]. We will also discuss how the results presented at the end of this thesis can be further leveraged to provide information about the gravitational waveform, and what other kinds of computational frameworks might serve to ease these calculations.

In addition to working with a quantum field theoretic description of gravity, several case-studies are worked through in the framework of scalar Quantum Electrodynamics which can be found in Appendices (A- H).

3.2. The 6-point tree level amplitude

3.2.1. The 3-Graviton Vertex and "Genuine" 3-Body Interaction

One of the features of general relativity is the ability for not only mass but energy to interact with gravity, with gravity itself included. Which is to say, gravity is able to self-interact. In a QFT description of gravity, this manifests as a vertex that couples three gravitons in a Feynman diagram.

Ina gauge theory such as gravity or Yang-Mills, individual Feynman diagrams shouldn't be taken to literally describe an interaction. This is because Feynman diagrams are drawn with gauge-fixed propagators and vertices, changing gauge might change the amplitude of a given diagram but observables (such as cross sections) will remain invariant as they contain contributions from all diagrams. (Of course, the classical potential V_{123} isn't gauge independent either, it is a tool used to calculate observables.) Still, the fact that these diagrams contribute at all means that we expect there to be some contribution from these 3-graviton vertices to the 3-body gravitational interaction, and thus there exist dynamics which are legitimately distinct from those identifiable by iterating over pairwise forces.

Given the ambiguity of gauge choices, we should consider how to define a "genuine" 3-body interaction in a way that is physically meaningful and independent of these choices. If we choose a convenient gauge, such that we can draw our diagram and calculate a full 6-point amplitude, the amplitude itself will be a gauge-independent observable (as it always is). In this way, the parts of this amplitude that correspond to the 3-graviton vertex can be isolated as the "true 3-body interaction."

The leading "genuine 3-body" potential arises from the classical limit of the tree-level $3 \to 3$ scattering amplitude of massive scalars [6, 7]. With this established, the primary objective of this calculation is to compute the full tree-level $3 \to 3$ gravitational scattering amplitude for three distinct massive scalars.

In order to solve for the 6-point amplitude in a gravity theory, we will first calculate the same amplitude in the context of scalar QCD and use the double copy to lift the expression to a description of gravity. We will work with an all-outgoing momentum convention, and a mostly-minus metric signature.

3.2.2. Calculating the Amplitude

To construct the 6-point tree level scalar QCD amplitude for three distinct massive scalars, we will leverage the modern approach of unitarity cuts, as seen in Figure 3.1).

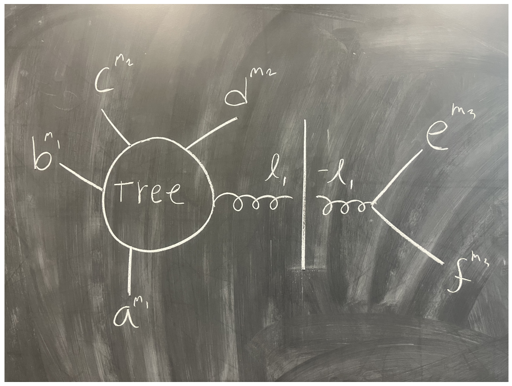

Figure 3.1. The cut 6-point tree level amplitude with three distinct massive scalars

The constituent expressions for this caluclation are the 5-point ordered amplitude for tree level scalar QCD, $A_{5,2}^{\text{tree}}(\ell_1, a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2})$ [7], and the 3-point scalar QCD tree amplitude.

$$
\begin{aligned}
A_{5,2}^{\text{tree}}(a, b^{m_1}, c^{m_1}, d^{m_2}, e^{m_2}) &= \frac{n_{5,2}^{M}(a,b,c,d,e)}{((a+b)^2 - m_1^2)(d+e)^2} + \frac{n_{5,2}^{M}(a,e,d,b,c)}{((a+e)^2 - m_2^2)(b+c)^2} + \frac{n^{\bar{M}_{5,2}(a,b,c,d,e)}}{(b+c)^2(d+e)^2} \\
A_{3,1}(a, b^{m_3}, c^{m_3}) &\sim k_b \cdot \varepsilon_a
\end{aligned}
$$

There are two distinct topologies at 5-point, one with only internal gluons and another which includes one internal scalar, they can be seen in Figure 3.2.

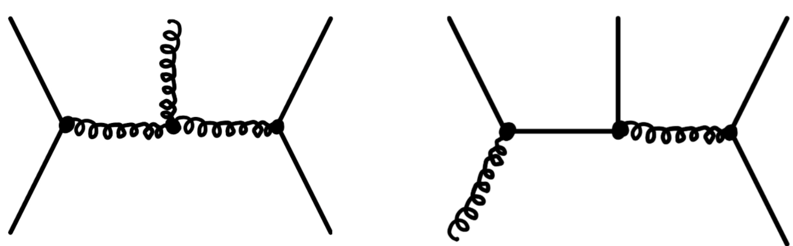

Figure 3.2. The two 5-point tree level topologies

The two 5-point topologies each contribute unique kinematic numerators

$$n_{5,2}^{\bar{M}}(a,b,c,d,e) = \frac{1}{4}\Big(2\big(t_{bc} + 2t_{cd} + t_{cc}\big)\big(t_{b\varepsilon_a} + t_{c\varepsilon_a}\big) + t_{ac}\big(t_{b\varepsilon_a} + 3t_{c\varepsilon_a} - 2t_{d\varepsilon_a}\big) + t_{ab}\big(t_{b\varepsilon_a} + 3t_{c\varepsilon_a} + 2t_{d\varepsilon_a}\big) + 4t_{ad}t_{c\varepsilon_a}\Big) \tag{3.1}$$

$$n_{5,2}^{M}(a,b,c,d,e) = \frac{1}{4}\Big(2t_{b\varepsilon_a}\big(t_{ac} + t_{bc} + 2t_{cd} + t_{cc}\big) + t_{ab}\big(t_{b\varepsilon_a} + t_{c\varepsilon_a} + 2t_{d\varepsilon_a}\big)\Big) \tag{3.2}$$

where $n^{\bar{M}}$ matches the 5-point topology with only internal gluons, and n^{M} matches the 5-point topology with an internal massive scalar.

Combining the amplitudes and summing over all states of the propagator with the projector operator, $P^{\mu\nu}(p,q) = \sum_{pols} \varepsilon^{\mu}(-p)\varepsilon^{\nu}(p) - \frac{q^{\mu}p^{\nu}+p^{\mu}q^{\nu}}{q\cdot p}$, yields the expression for the 6-point tree-level amplitude evaluated on the cuts.

$$\sum_{\text{states}} A_{5,2}^{\text{tree}}(\ell_1, a^{m_1}, b^{m_1}, e^{m_3}, f^{m_3})\ A_{3,1}^{\text{tree}}(c^{m_2}, d^{m_2}, -\ell_1) = A_{6,3}^{\text{tree}}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3})|_{\text{cuts}}$$

$$\begin{aligned} &= \Big\{t_{ae}^2 - t_{be}\big(2m_1^2 + 2t_{ab} + 4t_{ac} + t_{af} + t_{be} + t_{bf}\big) - 2\big(-t_{af} + t_{be} + t_{bf}\big)t_{ce} \\ &+ t_{ae}\big(2t_{ab} + t_{af} + 4t_{bc} + t_{bf} + 2(m_1^2 + t_{ce})\big)\Big\} \times \frac{1}{(4(k_a+k_b)^2(k_c+k_d)^2)} \\ &+ \frac{t_{ae}^2 + t_{af}\left(t_{be} + 2t_{ce}\right) + t_{ae}\big(2t_{ab} + t_{af} + 4t_{bc} + 3t_{be} + 2\big(m_1^2 + t_{bf} + t_{ce}\big)\big)}{4(k_c+k_d)^2\left(-m_1^2 + (k_a+k_e+k_f)^2\right)} \\ &+ \frac{t_{de}\big(4t_{ac} + 2t_{ae} + 2t_{cd} + 3t_{ce} + 2(m_2^2 + t_{cf}) + t_{de}\big) + (2t_{ae} + t_{ce} + t_{de})t_{df}}{4(k_a+k_b)^2\left(-m_2^2 + (k_d+k_e+k_f)^2\right)} \end{aligned} \tag{3.3}$$

Equation (3.3) is not the full 6-point scalar QCD tree-level amplitude, but is equivalent to the full amplitude with the cut conditions imposed. In order to solve for the full amplitude, Equation (3.3) will be compared to an ansatz of the full amplitude.

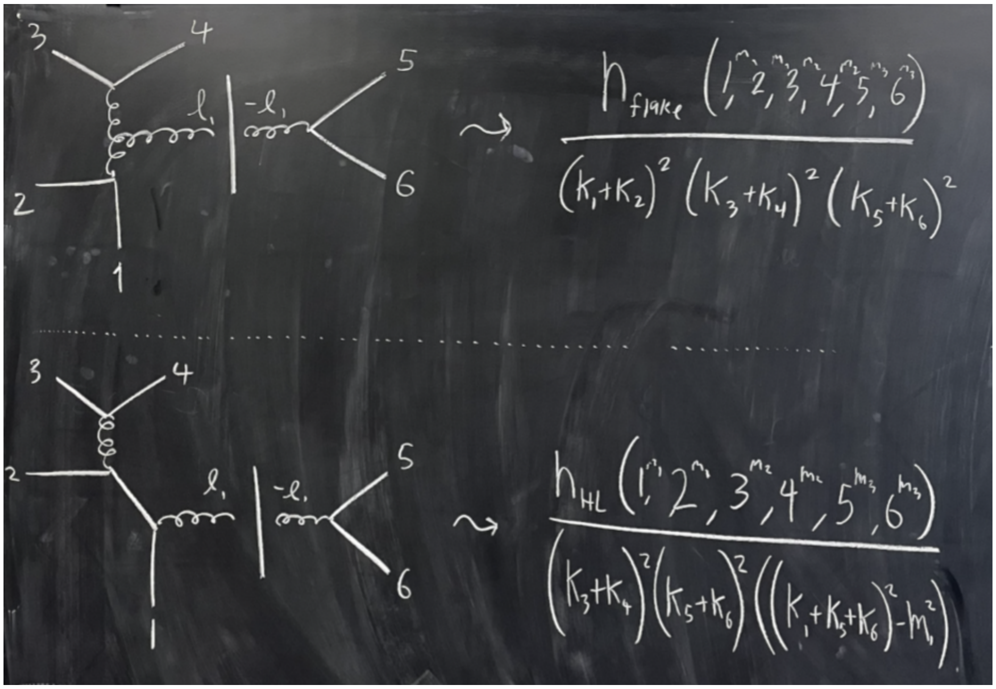

Figure 3.3. The 6 point topologies, and their contributions

Drawing out the possible topologies in Figure 3.3, we observe two distinct graph topologies: we'll call them the “snowflake" graph, which only includes gluon propagators, and the “half-ladder" graph, which includes a single massive scalar propagator. The numerator factors are distinct for each topology, but there are two distinct contributing forms of the half-ladder with different leg ordering. Including color, the full ansatz for the 6-point scalar QCD scattering amplitude for three massive scalars is:

$$
\begin{aligned}
A_{\text{ansatz}}&(a^{m_1},b^{m_1},c^{m_2},d^{m_2},e^{m_3},f^{m_3}) \\
&= \frac{c_{\text{flake}}(a^{m_1},b^{m_1},c^{m_2},d^{m_2},e^{m_3},f^{m_3})\;\; n_{\text{flake}}(a^{m_1},b^{m_1},c^{m_2},d^{m_2},e^{m_3},f^{m_3})}{(k_a+k_b)^2(k_c+k_d)^2(k_e+k_f)^2} \\
&+ \frac{c_{HL}(a^{m_1},b^{m_1},c^{m_2},d^{m_2},f^{m_3},e^{m_3})\;\; n_{HL}(a^{m_1},b^{m_1},c^{m_2},d^{m_2},f^{m_3},e^{m_3})}{(k_c+k_d)^2\,(k_a+k_b)^2\left(-k_e^2+(k_c+k_d+k_f)^2\right)} \\
&+ \frac{c_{HL}(e^{m_3},f^{m_3},d^{m_2},c^{m_2},a^{m_1},b^{m_1})\;\; n_{HL}(e^{m_3},f^{m_3},d^{m_2},c^{m_2},a^{m_1},b^{m_1})}{(k_c+k_d)^2\,(k_e+k_f)^2\left(-k_a^2+(k_c+k_d+k_b)^2\right)}
\end{aligned}
\tag{3.4}
$$

From here, we can build ansatze for the forms of n_{HL} and n_{flake} and solve by equating Eqs. (3.3) and (3.4). But what should these ansatze be? Because our graphs only contain massive scalars on external legs, we expect the numerator terms to be linear combinations of terms ($\alpha_i\ t_{a,b}\, t_{c,d}$) where α_i is some numerical coefficient, and a, b, c, d are external leg labelings, and t terms are dot products of external leg momenta, e.g.$t_{a,b} = (k_a \cdot k_b)$.

These ansatze are quite large since they each contain every possible pair of dot products with some coefficient. By using what we know the structure of the amplitude, we can solve for each α_i.

The first step is to match 6ptLHS6ptRHS, the cut condition, which carries an additional $\delta((k_3 + k_4)^2)$ momentum conservation rule due to the cut putting an internal gluonic leg on-shell. The rest of the relations are identifiable through automorphisms of the graphs:

$$\begin{aligned} n_{HL}(a,b,c,d,e,f) &= -\, n_{HL}(b,a,c,d,e,f) \\ &= -\, n_{HL}(a,b,d,c,e,f) \\ &= +\, n_{HL}(d,c,b,a,f,e) \end{aligned}$$

$$\begin{aligned} n_{\text{flake}}(a,b,c,d,e,f) &= -\, n_{\text{flake}}(b,a,c,d,e,f) \\ &= -\, n_{\text{flake}}(a,b,d,c,e,f) \\ &= -\, n_{\text{flake}}(a,b,c,d,f,e) \\ &= -\, n_{\text{flake}}(c,d,a,b,e,f) \\ &= -\, n_{\text{flake}}(e,f,c,d,a,b), \end{aligned} \tag{3.5}$$

and the Jacobi relation:

$$n_{\text{flake}}(a,b,c,d,e,f) = n_{HL}(a,b,c,d,e,f) - n_{HL}(a,b,c,d,f,e). \tag{3.6}$$

We are able to solve for the kinematic numerator under these constraints. The first leg ordering of the half-ladder contributes

$$
\begin{aligned}
n_{\mathrm{HL}}(k_1, k_2,&k_3, k_4, k_5, k_6) \\
&= m_1^2 \left(\frac{1}{8}(k_1 \cdot k_3) - \frac{1}{8}(k_1 \cdot k_4) - \frac{1}{8}(k_2 \cdot k_3) + \frac{1}{8}(k_2 \cdot k_4) - \frac{1}{4}(k_1 \cdot k_5) + \frac{1}{4}(k_2 \cdot k_5)\right) \\
&+ (k_1 \cdot k_5)\left(-\frac{5}{8}(k_2 \cdot k_3) + \frac{1}{8}(k_2 \cdot k_4) - \frac{1}{2}(k_3 \cdot k_5) - \frac{1}{4}(k_3 \cdot k_4) - \frac{m_3^2}{4}\right) \\
&+ \frac{1}{4}m_3^2(k_2 \cdot k_5) + (k_1 \cdot k_4)\left(-\frac{1}{8}(k_1 \cdot k_5) - \frac{1}{8}(k_2 \cdot k_5)\right) \\
&+ (k_1 \cdot k_2)\left(-\frac{1}{8}(k_1 \cdot k_4) - \frac{1}{8}(k_2 \cdot k_3) + \frac{1}{8}(k_2 \cdot k_4) - \frac{1}{4}(k_1 \cdot k_5) + \frac{1}{4}(k_2 \cdot k_5)\right) \\
&+ (k_1 \cdot k_3)\left(-\frac{3}{8}(k_1 \cdot k_5) + \frac{5}{8}(k_2 \cdot k_5) + \frac{1}{8}(k_1 \cdot k_2)\right) \\
&+ \frac{3}{8}(k_2 \cdot k_3)(k_2 \cdot k_5) + \frac{1}{8}(k_2 \cdot k_4)(k_2 \cdot k_5) + (k_2 \cdot k_5)\left(\frac{1}{2}(k_3 \cdot k_5) + \frac{1}{4}(k_3 \cdot k_4)\right) \\
&- \frac{1}{4}(k_1 \cdot k_5)^2 + \frac{1}{4}(k_2 \cdot k_5)^2, \qquad (3.7)
\end{aligned}
$$

where the remaining numerators can be found by swapping half-ladder leg orderings and applying jacobi. From here, the full 6-point scalar QCD scattering amplitude for three massive scalars can be constructed via 6ptLHS, with the numerator terms inserted, can be written out.

$$
\begin{aligned}
A^{6,3}_{\text{QCD}}(a^{m_1}, & b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3}) = \\
& \frac{c_{\text{flake}}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3})\ \ n_{\text{flake}}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3})}{(k_a + k_b)^2 (k_c + k_d)^2 (k_e + k_f)^2} \\
& + \frac{c_{HL}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, f^{m_3}, e^{m_3})\ \ n_{HL}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, f^{m_3}, e^{m_3})}{(k_c + k_d)^2\, (k_a + k_b)^2\, \left(-k_e^2 + (k_c + k_d + k_f)^2\right)} \\
& + \frac{c_{HL}(e^{m_3}, f^{m_3}, d^{m_2}, c^{m_2}, a^{m_1}, b^{m_1})\ \ n_{HL}(e^{m_3}, f^{m_3}, d^{m_2}, c^{m_2}, a^{m_1}, b^{m_1})}{(k_c + k_d)^2\, (k_e + k_f)^2\, \left(-k_a^2 + (k_c + k_d + k_b)^2\right)} \qquad (3.8)
\end{aligned}
$$

Equation (3.8) is converted into a $\mathcal{N} = 0$ supergravity amplitude via the double copy, resulting in Equation (3.9).

$$
\begin{aligned}
\mathcal{M}^{6,3}_{\text{gravity}}(a^{m_1}, & b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3}) = \\
& \frac{n_{\text{flake}}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3})\ \ n_{\text{flake}}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, e^{m_3}, f^{m_3})}{(k_a + k_b)^2 (k_c + k_d)^2 (k_e + k_f)^2} \\
& + \frac{n_{HL}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, f^{m_3}, e^{m_3})\ \ n_{HL}(a^{m_1}, b^{m_1}, c^{m_2}, d^{m_2}, f^{m_3}, e^{m_3})}{(k_c + k_d)^2\, (k_a + k_b)^2\, \left(-k_e^2 + (k_c + k_d + k_f)^2\right)} \\
& + \frac{n_{HL}(e^{m_3}, f^{m_3}, d^{m_2}, c^{m_2}, a^{m_1}, b^{m_1})\ \ n_{HL}(e^{m_3}, f^{m_3}, d^{m_2}, c^{m_2}, a^{m_1}, b^{m_1})}{(k_c + k_d)^2\, (k_e + k_f)^2\, \left(-k_a^2 + (k_c + k_d + k_b)^2\right)} \qquad (3.9)
\end{aligned}
$$

Thus, the 6-point tree level $\mathcal{N} = 0$ super gravity amplitude for three massive scalars has been identified. The final step in this investigation will be to see how this expression can be manipulated into classical observables of potential application to numerical astrophysical simulations, such as carried out in [10]. While corrections to the classical potential are the object of interest in many pre-existing implementations, there are complications with this approach which will be detailed at the end of this section. Instead, we will now investigate how this amplitude is directly related to observables such as momentum transfer.

3.3. The Eikonal Phase

Now the amplitude expression has been identified, we will go through the steps to extract classical observables in a way which could provide meaningful information for classical 3-body physics. The Eikonal phase is a gauge invariant classical observable that acts as a generating function for other observables, such as the scattering impulse, so that is where we will begin.

The Eikonal regime is valid when particle trajectories are scattered by only small angles. In this limit, particles paths are nearly unchanged by their interaction with the potential and the S-matrix takes on the simple form of a pure phase, $S \approx e^{i\chi}$.

This χ is the *Eikonal Phase*, the total phase accumulated by the particles wavefunction after the interaction. Classically, it is the classical action evaluated along an undeflected trajectory. Because χ is the action, it's derivatives produce physical observables - the derivative with respect to the impact parameter gives the impulse, the momentum deflection. In the following sections a systematic calculation of χ directly from the QFT scattering amplitude will be presented

3.3.1. Recasting Momenta

Since the Eikonal regime is valid for small momentum transfers, inorder to take the Eikonal limit the first step will be to recast this amplitude expression in terms of the average momenta of each particle, $\hat{p}_i^\mu$, and the momentum transfer (or "kick") the particle experiences during the scattering event, q_i^μ. They are defined as

$$\hat{p}_i = \frac{p_i^{\text{in}} + p_i^{\text{out}}}{2} \qquad , \qquad q_i = p_i^{\text{out}} - p_i^{\text{in}} \tag{3.10}$$

where $(\hat{p}_i)^2 \approx (m_i)^2$, and, due to the orthogonality condition, $\hat{p}_i \cdot q_i = 0$. This allows us to separate the large background kinematics from the small, scattering kinematics. By momentum conservation, $q_1 + q_2 + q_3 = 0$. This recasting is visualized in Figure 3.4, where notice we have changed from an all-ingoing to a half-in-half-out convention.

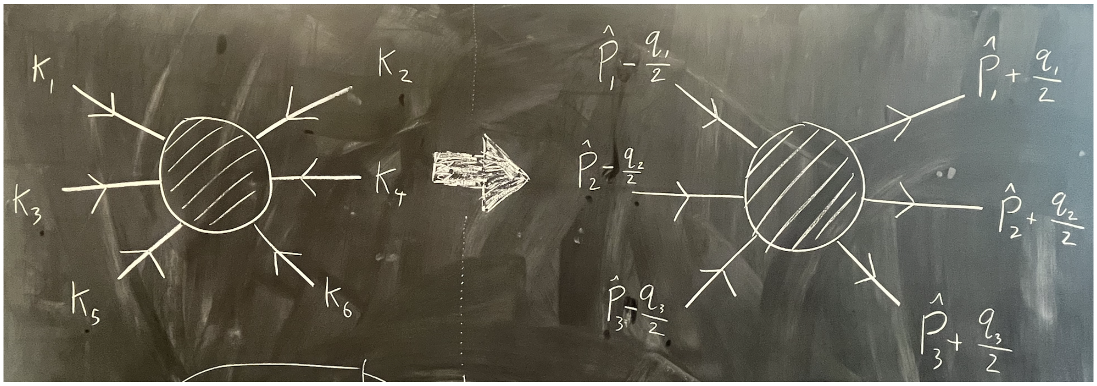

Figure 3.4. Recasting momentum as average $\hat{p}_i$ and transfer q_i (legs reordered for visualization)

When this exchange is carried out and conservation of momentum rules are applied, the numerator for the flake diagram, which knows about the "true 3-body interaction," becomes:

$$\begin{aligned} n_{\text{flake}} &\propto (\hat{p}_2 \cdot \hat{p}_3)(\hat{p}_1 \cdot q_3) + (\hat{p}_1 \cdot \hat{p}_3)(\hat{p}_2 \cdot q_1) + (\hat{p}_1 \cdot \hat{p}_2)(\hat{p}_3 \cdot q_2) \\ &\propto p_1^\mu p_2^\nu p_3^\rho \left(\eta_{\mu\rho}(q_1)_\nu + \eta_{\nu\mu}(q_2)_\rho + \eta_{\rho\nu}(q_3)_\mu\right) \\ &\propto p_1^\mu\, p_2^\nu\, p_3^\rho\, V_{\mu\nu\rho}(k_i) \end{aligned} \tag{3.11}$$

where $V_{\mu\nu\rho}(k_i)$ is the three-point all-gluon amplitude with the polarizations removed, which is to say the color-dual answer for the snowflake numerator is *literally* the same as connecting the average particle momenta to the three point amplitude.

But is this surprising? Indeed not, looking back at the structure of the snowflake it's clear to see how this happens. The massive scalar legs carry the average momenta of each particle, and the gluon leg carries the momentum kick q_i.

3.3.2. An Expansion into the Classical Limit

The by formally taking the limit where behaviors are large compared to $\hbar$, we isolate physics relevant to macroscopic bodies. In the Eikonal regime momentum transfer from the interaction is small and the classical limit is realized, which is to say actions are large compared to $\hbar$.

Let $\vec{P}_{cl,\,i}$ be the physical classical momentum of particle i. This momentum is macroscopic, measurable, and independent of $\hbar$. $\Delta\vec{P}_{cl,\,i}$ then is the classical physical impulse. The momentum transfer variable in the amplitude expression, $\vec{q}_i$, represents the momentum exchanged by the mediator. In the classical limit, the expectation value of this quantum momentum transfer will reproduce the classical impulse.

This connection is made carrying our a Fourier transform relating the impact parameter $\vec{b}$, which is classical, and the quantum momentum transfer. The phase of the corresponding wavefunction goes $e^{-\vec{q}\cdot\vec{b}/\hbar}$, and for the classical path to emerge the argument of the exponential must be of order 1 (by the stationary phase approximation. Since the impact parameter is macroscopic, we know that $\vec{q}_i \sim \mathcal{O}(\hbar)$ is requiered in the classical limit.

Therefore we say $\Delta\vec{P}_{cl,\,i} = \vec{q}_i/\hbar$, which remains finite as $|\Delta\vec{P}_{cl,\,i}| \gg \hbar$ is applied. From this discussion, we see why we will treat q_i as the small parameter in our expansion.

As an example, a massive propagator (with legs 3 & 4 between the propagator) in a diagram includes the denominator term

$$\frac{1}{(k_1+k_2+k_3)^2-k_3^2} = \frac{1}{-2\hat{p}_2\cdot q_3+(q_2+q_3)\cdot q_3} \to -\frac{1}{-2\hat{p}_2\cdot q_3\,\hbar} - \frac{(q_2+q_3)\cdot q_3}{4(\hat{p}_2\cdot q_3)^2} + \mathcal{O}(\hbar). \tag{3.12}$$

In the Eikonal limit where $\hat{p} \sim \mathcal{O}(\hbar^0)$ is large and $1_i \sim \mathcal{O}(\hbar^1)$ is small, this can be expanded in orders of $\hbar$. Keeping only the leading term gives the Eikonal propagator:

$$\frac{1}{2\hat{p}_i \cdot q_j}. \tag{3.13}$$

3.3.2.1. The Classical Charge Limit. The second part of taking the classical limit involves the couplings.

Classically, we consider classical sources to have large charges. This means large color charge operator eigenvalues for Yang-Mills, or large mass for gravity. In the limit of large charge, the non-Abelian commutator becomes sub-leading to the symmetric product of charges and the thus the quantum color algebra simplifies greatly. For tree level calculation, this means that the quantum color operators, or Lie algebra generators, become classical color vectors ($T^a \to c^a$). Thus, products also simplify $T^a T^b \to c^a c^b$. Additionally, the structure constant, which represents the non-commutativity algebra, is suppressed by $\hbar\ f^{abc} \to \mathcal{O}(\hbar)$.

The suppression of the structure constant implies that diagram topologies involving the 3-gluon vertex, such as the snowflake, are formally quantum corrections to the leading classical Eikonal phase in Yang-Mills theory and can be dropped at this order. This is not the case with gravity, and the corresponding 3-graviton vertex provides a leading genuine 3-body force.

3.3.3. Setting Up Extracting the Eikonal Phase from the Amplitude

Now we apply the above limits to our expression for the amplitude, recasting momentum and taking the limit of small q_i. When we apply the classical charge limit the snowflake diagram's contribution vanishes, and we are left the classical part of the amplitude, which we'll call $\mathcal{A}^{\text{tree, classical}}$. In this final section, we will connect this to the eikonal phase χ from the beginning of the discussion.

3.3.3.1. On-Shell Constraints. By modifying the on-shell condition $2\hat{p}_i \cdot q_i + q_i^2 = 0$ for small momentum transfer we arrive at the approximation $2\hat{p}_i \cdot q_i \approx 0$, for leading classical order.

This can be made more rigorous by recalling $q_i \sim \mathcal{O}(\hbar)$. Our expression is then of the form $\mathcal{O}(\hbar) + \mathcal{O}(\hbar^2) = 0$, and since each order of $\hbar$ must vanish independently we can further enforce that $2\hat{p}_i \cdot q_i \approx 0$.

3.3.3.2. The Integration Measure $d\mu_N$. The full measure for the N=body ekional phase is an integral over all possible momentym transfers, subject to the eikonal constraints and overall momentum conservation. For the $N = 3$ case (3-body), this is written as:

$$\int d\mu_3 = \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \\ \times (2\pi)^D \delta^D(q_1 + q_2 + q_3)(2\pi)\delta(2p_1 \cdot q_1)(2\pi)\delta(2p_2 \cdot q_2)(2\pi)\delta(2p_3 \cdot q_3) \tag{3.14}$$

One of the Eikonal limit delta functions is redundant, from momentum conservation.

3.3.3.3. Introducing Impact Parameters. Classical scattering is described by trajectories with well-defined impact parameters, which are vectors describing the geometry of the event. impact parameter of particle i relative to particle j is $\vec{b}_{i\,j}$, for three particles $\vec{b}_{12} = \vec{b}_{13} - \vec{b}_{23}$. We see that for three particles, the full dynamics are encoded with two impact parameter vectors. Since the impact parameter is the Fourier conjugate to momentum transfer, we are able to express our momentum-space amplitude in terms of the physical impact parameters by performing a Fourier transform and express the eikonal phase integral χ as a function of the impact parameters.

$$\chi(\vec{b}_{13}, \vec{b}_{23}) = \mathcal{N} \int d^{D-2}\vec{q}_{1\perp} d^{D-2}\vec{q}_{2\perp} e^{-i(\vec{q}_{1\perp}\cdot\vec{b}_{13} + \vec{q}_{2\perp}\cdot\vec{b}_{23})} \mathcal{A}_{\text{eikonal}}(\vec{q}_{1\perp}, \vec{q}_{2\perp}) \tag{3.15}$$

where $\mathcal{A}_{\text{eikonal}}$ is the amplitude evaluated in the Eikonal limit.

3.3.4. The Irreducible part of the amplitude

Of particular interest is isolating the “genuine 3-body" (or irreducible) part of the interaction, since these are the source of a the non-pairwise (thus “genuine 3-body") contribution to the interaction. This only accounts for one piece of the full $3 \to 3$ amplitude with the other piece being the “factorizable" part, carrying information about pairwise interactions.

Both of these are physical. Imagine the topology of a factorizable graph, wherein particle 1 scatters off of particle 2, and then particle 2 scatters off particle 3 – this is the half ladder topology! Classically, we might imagine that this physically represents a situation where the 3-body force is subleading to individual interactions. This is different than the simple sum of the two scattering events because of the internal massive propagator belonging to particle 2.

As theoretical physicists trying to understand the structure of gravity, this irreducible part represents genuinely new information we couldn't have just guessed by knowing the 2-body Newtonian force law! It describes dynamics not included in Newtonian gravity, and provides us with fundamental new information.

As far as approach to solving goes, there are two obvious choices. One could calculate the full $3/to3$ amplitude before identifying and subtracting off the factorizable part of the expression, or one could identify a method which isolates the irreducible 3-body dynamics from the beginning. From an amplitudes perspective, the strategy of isolating the 3-body part early is strongly preferable since it will be much easier to calculate this amplitude than to identify and subtract the pair-wise interaction terms in the full expression.

3.3.5. From the 3-Body Eikonal Phase to Physical Observables

The final step is to use the generating function defined above to compute concrete, physical observables from the new physics.

3.3.6. The Impulse

The most direct observable is the impulse on each particle in the interaction, which is the total change in the particle's momentum. This observable is also of potential direct use for perdictive models of these types of interactions. In our current chosen coordinate system, the eikonal phase is a function relative to the impact parameters $\vec{b}_{13}, \vec{b}_{23}$, and impulses are their conjugate variables. They are related as

$$\Delta\vec{p}_{1\perp} = -\frac{\partial\chi_2}{\partial\vec{b}_{13}} \qquad , \qquad \Delta\vec{p}_{2\perp} = -\frac{\partial\chi_2}{\partial\vec{b}_{23}} \tag{3.16}$$

where $\perp$ reminds us that we are considering components of impulse transverse to the direction of motion. The third impulse component is easily identified by conservation of momentum, $\Delta\vec{p}_{1\perp} + \Delta\vec{p}_{2\perp} + \Delta\vec{p}_{3\perp} = 0$. By calculating the scalar function χ_2, we have implicitly calculated the impulse for all three particles.

3.4. Future Directions

3.4.1. The Gravitational Waveform

The Eikonal phase describes the conservative part of the interaction. To calculate information about energy lost to radiation, the on-shell amplitudes toolkit can be used to compute the amplitude for emitting a real graviton. In this case, the fundamental object is the 7-point tree level amplitude $\mathcal{A}_{3\to3+g}(p_1, ...p_6; k)$ where k is the on-shell momentum of the emitted graviton, so $k^2 = 0$. A theorem by Weinberg [14] shows that in the soft limit of the emitted graviton, with $k \to 0$, this amplitude has a universal, factorized structure which is obtained by multiplying the 6-point amplitude by a universal "soft factor":

$$\mathcal{A}_{3\to3+g} \to \left(\sim_{i=1}^{6} \frac{\varepsilon_{\mu\nu}p_i^\mu p_i^\nu}{p_i \cdot k}\right) \times \mathcal{A}_{3\to3} \tag{3.17}$$

where $\varepsilon_{\mu\nu}$ is the polarization tensor of the graviton. There is an established direct link between this single-graviton emission and the classical gravitational waveform, and the classical radiated field can be constructed from a Fourier transform of the on-shell amplitude. Example calculations of radiation emission can be found in Appendix D, and Appendix G demonstrates the kind of analysis which can be done to the gravitational waveform in the context of scalar QED, where a photon emitted by a bound state transition is analyzed in terms of its angular momentum.

Here is the power of working in this on-shell framework. Beginning with on-shell amplitudes as building blocks, we can recreate descriptions of the conservative forces which govern scattering processes and provide analytical predictions of the emitted classical gravitational waveforms from these processes.

3.4.2. 3-Body EFTs

Depending on the problem at hand, it may be useful to consider additional simplifications to the problem of 3-body scattering and work within different EFTs. For example, one such simplification could be considering the black hole binary as a closed bound state instead of a pair of free particles. Appendix E contains details on how bound states arise in the scattering matrix, Appendix F investigates matching bound state transition matrix elements to an EFT, and Appendix I contains further discussion of what a 3-body EFT might look like, and how one would be constructed and matched to the full theory.

3.4.3. Corrections to the Classical Potential

In some already existing implementations of numerical simulations for astrophysical phenomena, corrections to the classical position space potential may prove to be more directly useful than classical observables such as the impulse. As was noted before, potentials are related directly to amplitudes by Fourier transforms, and so the gravitational amplitude found in Equation (3.9) can be transformed directly into an expression for the classical potential.

Of particular interest to this discussion is the part of the potential which originates directly from the non-factorizable part of the amplitude – the “genuine" 3-body interaction. Thus, isolating this part of the potential may prove to be the next major step in this investigation.

The first way this might be accomplished in general would be to follow the recent work of Solon and Wolz [13], in which the non-factorizable part of the potential was isolated from the full scattering amplitude via a series of limits and Fourier transforms. This method proves to be computationally expensive, introducing non-physical divergences which must be taken into account, but does produce the desired result.

A simpler path may be to use the worldline formalism which centers on an effective action, S_{eff}, whose integrand describes classical trajectories [12]. Because the amplitude and worldline pictures must provide the same result for the same physical situation, this framework can be used to isolate the irreducible parts of the amplitude expression in Equation (3.9) into the effective action and the classical trajectories and position-space potential can be extracted. This is an extremely promising future direction for this research, as it is less computationally expensive than the previously noted approach. Issues that arise along this path should be resolvable by leveraging the Lippman-Schwinger equation, for which further details can be found in Appendix C.

CHAPTER 4

Conclusion

In this thesis we have demonstrated that classical limits of observables in a QFT description of gravity can be used to provide useful and potentially unique information for astrophysical numerical simulations of 3-body black hole gravitational scattering. Specifically, we have seen that by constructing a scalar $\mathcal{N} = 0$ supergravity scattering amplitude via unitarity methods and the double copy, we are able to impose the Eikonal limit and successfully retrieve the Eikonal phase as a generating function for classical observables.

The success of this method hints at various future directions for this work such as analyzing gravitational waveforms, constructing 3-body EFTs, and directly calculating corrections to the classical potential originating from the irreducible parts of the 6-point scattering amplitude for 3 distinct massive scalar particles. This last path is particularly exciting, as the use of the worldline formalism EFT to more efficiently obtain results which were previously computationally expensive is an intriguing prospect.

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